If there are more than two regions which are approached by an odd number of bridges, no route satisfying the required conditions can be found.

If, however, there are only two regions with an odd number of approach bridges the required journey can be completed provided it originates in one of the regions.

If, finally, there is no region with an odd number of approach bridges, the required journey can be effected, no matter where it begins.

These rules solve completely the problem initially proposed.

21. After we have determined that a route actually exists we are left with the question how to find it. To this end the following rule will serve: Wherever possible we mentally eliminate any two bridges that connect the same two regions; this usually reduced the number of bridges considerably. Then—and this should not be difficult—we proceed to trace the required route across the remaining bridges. The pattern of this route, once we have found it, will not be substantially affected by the restoration of the bridges which were first eliminated from consideration—as a little thought will show; therefore I do not think I need say more about finding the routes themselves.

I must go in and out. —BERNARD SHAW (*Heartbreak House*)

5 Topology

By RICHARD COURANT and HERBERT ROBBINS

EULER'S FORMULA FOR POLYHEDRA

ALTHOUGH the study of polyhedra held a central place in Greek geometry, it remained for Descartes and Euler to discover the following fact: In a simple polyhedron let V denote the number of vertices, E the number of edges, and F the number of faces; then always

$$V - E + F = 2. \tag{1}$$

By a *polyhedron* is meant a solid whose surface consists of a number of polygonal faces. In the case of the regular solids, all the polygons are congruent and all the angles at vertices are equal. A polyhedron is *simple* if there are no "holes" in it, so that its surface can be deformed continuously into the surface of a sphere. Figure 2 shows a simple polyhedron which is not regular, while Figure 3 shows a polyhedron which is not simple.

The reader should check the fact that Euler's formula holds for the simple polyhedra of Figures 1 and 2, but does not hold for the polyhedron of Figure 3.

To prove Euler's formula, let us imagine the given simple polyhedron to be hollow, with a surface made of thin rubber. Then if we cut out one of the faces of the hollow polyhedron, we can deform the remaining surface until it stretches out flat on a plane. Of course, the areas of the faces and the angles between the edges of the polyhedron will have changed in this process. But the network of vertices and edges in the plane will contain the same number of vertices and edges as did the original polyhedron, while the number of polygons will be one less than in the original polyhedron, since one face was removed. We shall now show that for the plane network, $V - E + F = 1$, so that, if the removed face is counted, the result is $V - E + F = 2$ for the original polyhedron.

First we "triangulate" the plane network in the following way: In some polygon of the network which is not already a triangle we draw a diagonal. The effect of this is to increase both E and F by 1, thus pre-

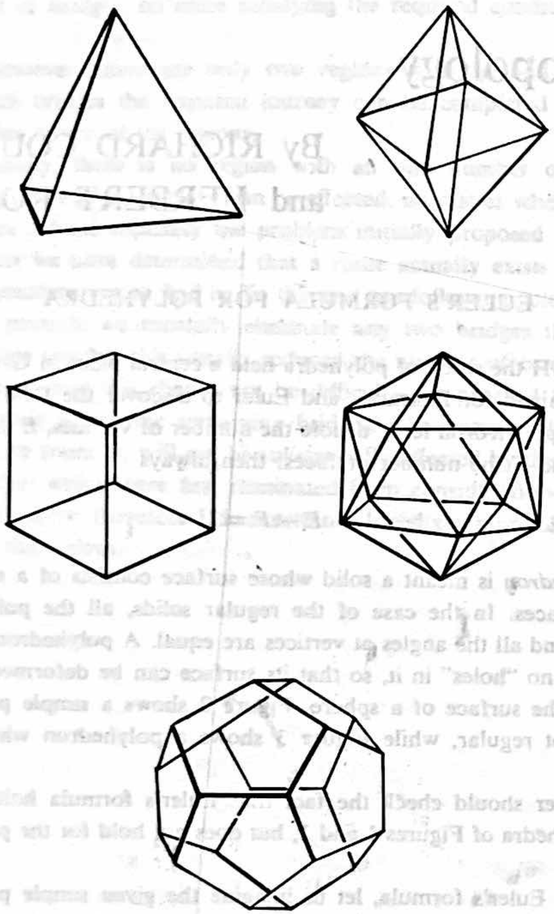

FIGURE 1—The regular polyhedra.

serving the value of $V - E + F$. We continue drawing diagonals joining pairs of points (Figure 4) until the figure consists entirely of triangles, as it must eventually. In the triangulated network, $V - E + F$ has the value that it had before the division into triangles, since the drawing of diagonals has not changed it. Some of the triangles have edges on the boundary of the plane network. Of these some, such as ABC, have only one edge on the boundary, while other triangles may have two edges on the boundary. We take any boundary triangle and remove that part of it which does not also belong to some other triangle. Thus, from ABC we remove the edge AC and the face, leaving the vertices A, B, C and the two edges AB and BC; while from DEF we remove the face, the two

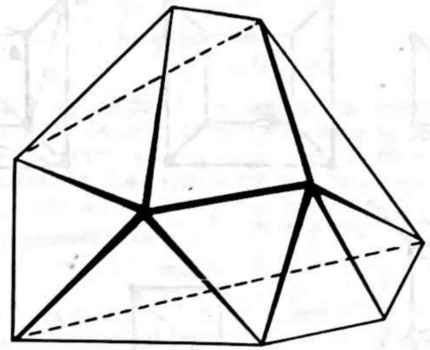

FIGURE 2—A simple polyhedron. $V - E + F = 9 - 18 + 11 = 2$.

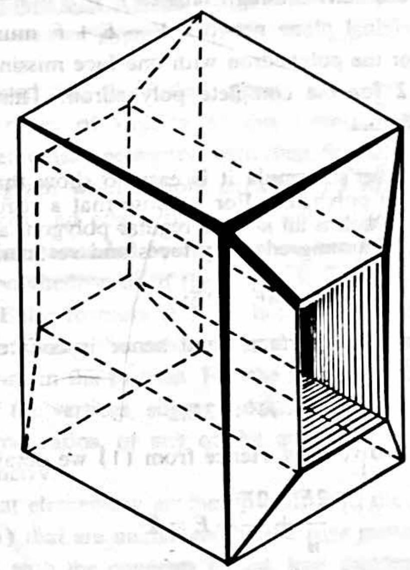

FIGURE 3—A non-simple polyhedron. $V - E + F = 16 - 32 + 16 = 0$.

edges DF and FE, and the vertex F. The removal of a triangle of type ABC decreases E and F by 1, while V is unaffected, so that $V - E + F$ remains the same. The removal of a triangle of type DEF decreases V by 1, E by 2, and F by 1, so that $V - E + F$ again remains the same. By a properly chosen sequence of these operations we can remove triangles with edges on the boundary (which changes with each removal), until finally only one triangle remains, with its three edges, three vertices, and one face. For this simple network, $V - E + F = 3 - 3 + 1 = 1$. But we

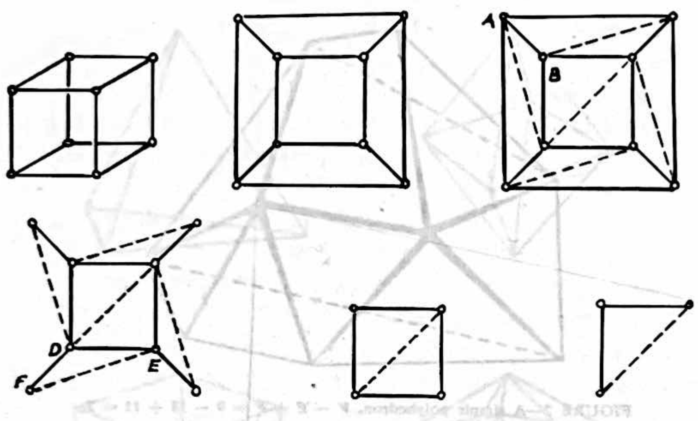

FIGURE 4—Proof of Euler's theorem.

have seen that by constantly erasing triangles $V-E+F$ was not altered. Therefore in the original plane network $V-E+F$ must equal 1 also, and thus equals 1 for the polyhedron with one face missing. We conclude that $V-E+F=2$ for the complete polyhedron. This completes the proof of Euler's formula.

On the basis of Euler's formula it is easy to show that there are no more than five regular polyhedra. For suppose that a regular polyhedron has F faces, each of which is an n-sided regular polygon, and that r edges meet at each vertex. Counting edges by faces and vertices, we see that

$$nF = 2E; \tag{2}$$

for each edge belongs to two faces, and hence is counted twice in the product nF; moreover,

$$rV = 2E, \tag{3}$$

since each edge has two vertices. Hence from (1) we obtain the equation

$$\frac{2E}{n}+\frac{2E}{r}-E=2$$

or

$$\frac{1}{n}+\frac{1}{r}=\frac{1}{2}+\frac{1}{E}. \tag{4}$$

We know to begin with that $n \geq$ and $r \geq 3$, since a polygon must have at least three sides, and at least three sides must meet at each polyhedral angle. But n and r cannot both be *greater* than three, for then the left hand side of equation (4) could not exceed ½, which is impossible for any positive value of E. Therefore, let us see what values r may have when $n=3$, and what values n may have when $r=3$. The totality of polyhedra given by these two cases gives the number of possible regular polyhedra.

For $n=3$, equation (4) becomes

$$\frac{1}{r}-\frac{1}{6}=\frac{1}{E};$$

r can thus equal 3, 4, or 5. (6, or any greater number, is obviously excluded, since $1/E$ is always positive.) For these values of n and r we get $E=6$, 12, or 30, corresponding respectively to the tetrahedron, octahedron, and icosahedron. Likewise, for $r=3$ we obtain the equation

$$\frac{1}{n}-\frac{1}{6}=\frac{1}{E},$$

from which it follows that $n=3$, 4, or 5, and $E=6$, 12, or 30, respectively. These values correspond respectively to the tetrahedron, cube, and dodecahedron. Substituting these values for n, r, and E in equations (2) and (3), we obtain the numbers of vertices and faces in the corresponding polyhedra.

TOPOLOGICAL PROPERTIES OF FIGURES

TOPOLOGICAL PROPERTIES

We have proved that the Euler formula holds for any simple polyhedron. But the range of validity of this formula goes far beyond the polyhedra of elementary geometry, with their flat faces and straight edges; the proof just given would apply equally well to a simple polyhedron with curved faces and edges, or to any subdivision of the surface of a sphere into regions bounded by curved arcs. Moreover, if we imagine the surface of the polyhedron or of the sphere to be made out of a thin sheet of rubber, the Euler formula will still hold if the surface is deformed by bending and stretching the rubber into any other shape, so long as the rubber is not torn in the process. For the formula is concerned only with the *numbers* of the vertices, edges, and faces, and not with lengths, areas, straightness, cross-ratios, or any of the usual concepts of elementary or projective geometry.

We recall that elementary geometry deals with the magnitudes (length, angle, and area) that are unchanged by the rigid motions, while projective geometry deals with the concepts (point, line, incidence, and cross-ratio) which are unchanged by the still larger group of projective transformations. But the rigid motions and the projections are both very special cases of what are called *topological transformations:* a topological transformation of one geometrical figure A into another figure A' is given by any correspondence

$$p \leftrightarrow p'$$

between the points p of A and the points p' of A' which has the following two properties:

1. *The correspondence is biunique.* This means that to each point p of A corresponds just one point p' of A', and conversely.

2. *The correspondence is continuous in both directions.* This means that if we take any two points p, q of A and move p so that the distance between it and q approaches zero, then the distance between the corresponding points p', q' of A' will also approach zero, and conversely.

Any property of a geometrical figure A that holds as well for every figure into which A may be transformed by a topological transformation is called a *topological property* of A, and *topology* is the branch of geometry which deals only with the topological properties of figures. Imagine a figure to be copied "free-hand" by a conscientious but inexpert draftsman who makes straight lines curved and alters angles, distances and areas; then, although the metric and projective properties of the original figure would be lost, its topological properties would remain the same.

The most intuitive examples of general topological transformations are the *deformations.* Imagine a figure such as a sphere or a triangle to be made from or drawn upon a thin sheet of rubber, which is then stretched and twisted in any manner without tearing it and without bringing distinct points into actual coincidence. (Bringing distinct points into coincidence would violate condition 1. Tearing the sheet of rubber would violate condition 2, since two points of the original figure which tend toward coincidence from opposite sides of a line along which the sheet is torn would not tend towards coincidence in the torn figure.) The final position of the figure will then be a topological image of the original. A triangle can be deformed into any other triangle or into a circle or an ellipse, and hence these figures have exactly the same topological properties. But one

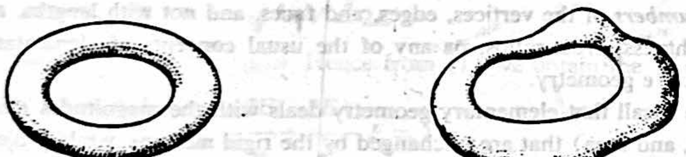

FIGURE 5—Topologically equivalent surfaces.

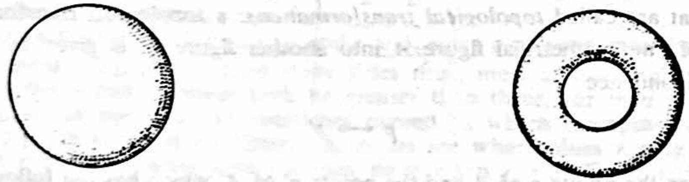

FIGURE 6—Topologically non-equivalent surfaces.

cannot deform a circle into a line segment, nor the surface of a sphere into the surface of an inner tube.

The general concept of topological transformation is wider than the concept of deformation. For example, if a figure is cut during a deformation and the edges of the cut sewn together after the deformation in exactly the same way as before, the process still defines a topological transformation of the original figure, although it is not a deformation. Thus the two curves of Figure 12 [1] are topologically equivalent to each other or to a circle, since they may be cut, untwisted, and the cut sewn up. But it is impossible to deform one curve into the other or into a circle without first cutting the curve.

Topological properties of figures (such as are given by Euler's theorem and others to be discussed in this section) are of the greatest interest and importance in many mathematical investigations. They are in a sense the deepest and most fundamental of all geometrical properties, since they persist under the most drastic changes of shape.

CONNECTIVITY

As another example of two figures that are not topologically equivalent we may consider the plane domains of Figure 7. The first of these consists

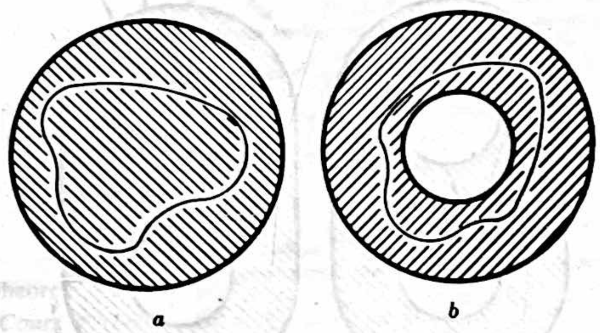

FIGURE 7—Simple and double connectivity.

of all points interior to a circle, while the second consists of all points contained between two concentric circles. Any closed curve lying in the domain a can be continuously deformed or "shrunk" down to a single point *within the domain.* A domain with this property is said to be *simply connected.* The domain b is not simply connected. For example, a circle concentric with the two boundary circles and midway between them cannot be shrunk to a single point within the domain, since during this process the curve would necessarily pass over the center of the circles, which is not a point of the domain. A domain which is not simply connected is said to be *multiply connected.* If the multiply connected domain

[1] [See p. 592, ED.]

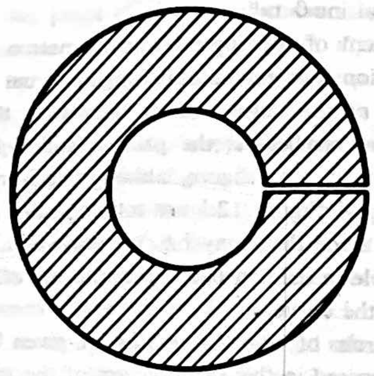

FIGURE 8—Cutting a doubly connected domain to make it simply connected.

b is cut along a radius, as in Figure 8, the resulting domain is simply connected.

More generally, we can construct domains with two, three, or more "holes," such as the domain of Figure 9. In order to convert this domain into a simply connected domain, two cuts are necessary. If $n-1$ non-

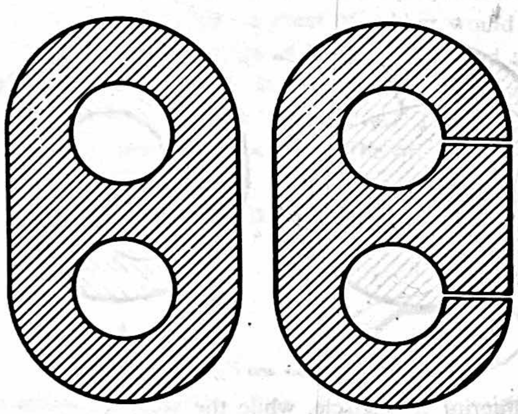

FIGURE 9—Reduction of a triply connected domain.

intersecting cuts from boundary to boundary are needed to convert a given multiply connected domain *D* into a simply connected domain, the domain *D* is said to be *n*-tuply connected. The degree of connectivity of a domain in the plane is an important topological invariant of the domain.

OTHER EXAMPLES OF TOPOLOGICAL THEOREMS

THE JORDAN CURVE THEOREM

A simple closed curve (one that does not intersect itself) is drawn in the plane. What property of this figure persists even if the plane is

regarded as a sheet of rubber that can be deformed in any way? The length of the curve and the area that it encloses can be changed by a deformation. But there is a topological property of the configuration which is so simple that it may seem trivial: *A simple closed curve C in the plane divides the plane into exactly two domains, an inside and an outside.* By this is meant that the points of the plane fall into two classes—*A*, the outside of the curve, and *B*, the inside—such that any pair of points of the same class can be joined by a curve which does not cross *C*, while any curve joining a pair of points belonging to different classes must cross *C*. This statement is obviously true for a circle or an ellipse, but the self-evidence fades a little if one contemplates a complicated curve like the twisted polygon in Figure 10.

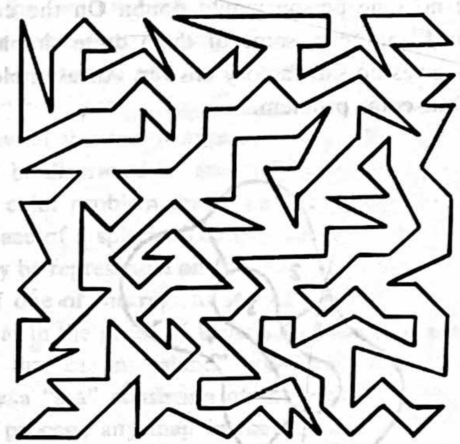

FIGURE 10—Which points of the plane are inside this polygon?

This theorem was first stated by Camille Jordan (1838–1922) in his famous *Cours d'Analyse*, from which a whole generation of mathematicians learned the modern concept of rigor in analysis. Strangely enough, the proof given by Jordan was neither short nor simple, and the surprise was even greater when it turned out that Jordan's proof was invalid and that considerable effort was necessary to fill the gaps in his reasoning. The first rigorous proofs of the theorem were quite complicated and hard to understand, even for many well-trained mathematicians. Only recently have comparatively simple proofs been found. One reason for the difficulty lies in the generality of the concept of "simple closed curve," which is not restricted to the class of polygons or "smooth" curves, but includes all curves which are topological images of a circle. On the other hand, many concepts such as "inside," "outside," etc., which are so clear to the intuition, must be made precise before a rigorous proof is possible. It is of the highest theoretical importance to analyze such concepts in their fullest generality, and much of modern topology is devoted to this task. But one should never forget that in the great majority of cases that arise from the study of concrete geometrical phenomena it is quite beside the point to work with concepts whose extreme generality creates unnecessary difficulties. As a matter of fact, the Jordan curve theorem is quite simple to prove for the reasonably well-behaved curves, such as polygons or curves with continuously turning tangents, which occur in most important problems.

THE FOUR COLOR PROBLEM

From the example of the Jordan curve theorem one might suppose that topology is concerned with providing rigorous proofs for the sort of obvious assertions that no sane person would doubt. On the contrary, there are many topological questions, some of them quite simple in form, to which the intuition gives no satisfactory answer. An example of this kind is the renowned "four color problem."

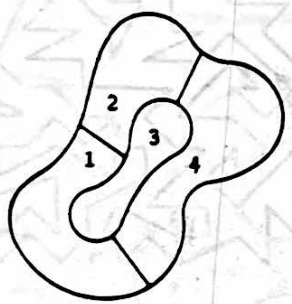

FIGURE 11—Coloring a map.

In coloring a geographical map it is customary to give different colors to any two countries that have a portion of their boundary in common. It has been found empirically that any map, no matter how many countries it contains nor how they are situated, can be so colored by using only *four* different colors. It is easy to see that no smaller number of colors will suffice for all cases. Figure 11 shows an island in the sea that certainly cannot be properly colored with less than four colors, since it contains four countries, each of which touches the other three.

The fact that no map has yet been found whose coloring requires more than four colors suggests the following mathematical theorem: *For any subdivision of the plane into non-overlapping regions, it is always possible to mark the regions with one of the numbers* 1, 2, 3, 4 *in such a way that no two adjacent regions receive the same number.* By "adjacent" regions we mean regions with a whole segment of boundary in common; two regions which meet at a single point only or at a finite number of points (such as the states of Colorado and Arizona) will not be called adjacent, since no confusion would arise if they were colored with the same color.

The problem of proving this theorem seems to have been first proposed by Moebius in 1840, later by DeMorgan in 1850, and again by Cayley in 1878. A "proof" was published by Kempe in 1879, but in 1890 Heawood found an error in Kempe's reasoning. By a revision of Kempe's proof, Heawood was able to show that *five* colors are always sufficient. Despite the efforts of many famous mathematicians, the matter essentially rests with this more modest result: It has been *proved* that five colors suffice for all maps and it is *conjectured* that four will likewise suffice. But, as in the case of the famous Fermat theorem neither a proof of this conjecture nor an example contradicting it has been produced, and it remains one of the great unsolved problems in mathematics. The four color theorem has indeed been proved for all maps containing less than thirty-eight regions. In view of this fact it appears that even if the general theorem is false it cannot be disproved by any very simple example.

In the four color problem the maps may be drawn either in the plane or on the surface of a sphere. The two cases are equivalent: any map on the sphere may be represented on the plane by boring a small hole through the interior of one of the regions A and deforming the resulting surface until it is flat, as in the proof of Euler's theorem. The resulting map in the plane will be that of an "island" consisting of the remaining regions, surrounded by a "sea" consisting of the region A. Conversely, by a reversal of this process, any map in the plane may be represented on the sphere. We may therefore confine ourselves to maps on the sphere. Furthermore, since deformations of the regions and their boundary lines do not affect the problem, we may suppose that the boundary of each region is a simple closed polygon composed of circular arcs. Even thus "regularized," the problem remains unsolved; the difficulties here, unlike those involved in the Jordan curve theorem, do not reside in the generality of the concepts of region and curve.

A remarkable fact connected with the four color problem is that for surfaces more complicated than the plane or the sphere the corresponding theorems have actually been proved, so that, paradoxically enough, the analysis of more complicated geometrical surfaces appears in this respect to be easier than that of the simplest cases. For example, on the surface of a torus (see Figure 5), whose shape is that of a doughnut or an inflated inner tube, it has been shown that any map may be colored by using seven colors, while maps may be constructed containing seven regions, each of which touches the other six.

KNOTS

As a final example it may be pointed out that the study of knots presents difficult mathematical problems of a topological character. A knot is formed by first looping and interlacing a piece of string and then joining the ends together. The resulting closed curve represents a geometrical figure that remains essentially the same even if it is deformed by pulling or twisting without breaking the string. But how is it possible to give an intrinsic characterization that will distinguish a knotted closed curve in space from an unknotted curve such as the circle? The answer is by no means simple, and still less so is the complete mathematical analysis of the various kinds of knots and the differences between them. Even for the simplest case this has proved to be a sizable task. Consider the two trefoil knots shown in Figure 12. These two knots are completely symmetrical "mirror images" of one another, and are topologically equivalent, but they are not congruent. The problem arises whether it is possible to deform one of these knots into the other in a continuous way. The answer is in the negative, but the proof of this fact requires considerably more knowledge of the technique of topology and group theory than can be presented here.

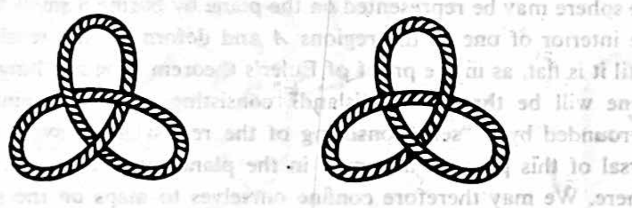

FIGURE 12—Topologically equivalent knots that are not deformable into one another.

THE TOPOLOGICAL CLASSIFICATION OF SURFACES

THE GENUS OF A SURFACE

Many simple but important topological facts arise in the study of two-dimensional surfaces. For example, let us compare the surface of a sphere with that of a torus. It is clear from Figure 13 that the two surfaces differ in a fundamental way: on the sphere, as in the plane, every simple closed curve such as C separates the surface into two parts. But on the torus there exist closed curves such as C' that do not separate the surface into two parts. To say that C separates the sphere into two parts means that if the sphere is cut along C it will fall into two distinct and unconnected pieces, or, what amounts to the same thing, that we can find two points on the sphere such that any curve on the sphere which joins them must intersect C. On the other hand, if the torus is cut along the closed curve

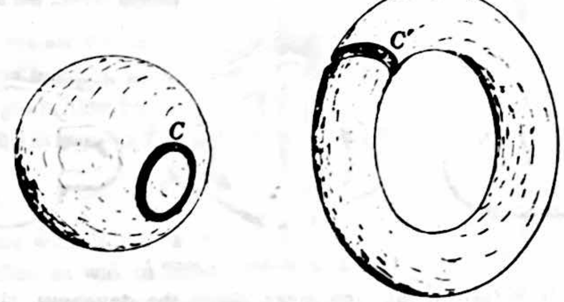

FIGURE 13—Cuts on sphere and torus.

C', the resulting surface still hangs together: any point of the surface can be joined to any other point by a curve that does not intersect C'. This difference between the sphere and the torus marks the two types of surfaces as topologically distinct, and shows that it is impossible to deform one into the other in a continuous way.

Next let us consider the surface with two holes shown in Figure 14. On this surface we can draw *two* non-intersecting closed curves A and B which do not separate the surface. The torus is always separated into two parts by any two such curves. On the other hand, *three* closed non-intersecting curves always separate the surface with two holes.

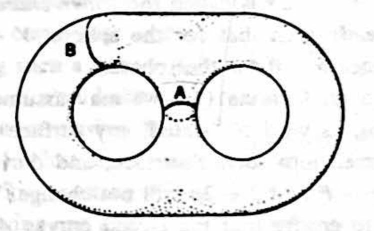

FIGURE 14—A surface of genus 2.

These facts suggest that we define the *genus* of a surface as the largest number of non-intersecting simple closed curves that can be drawn on the surface without separating it. The genus of the sphere is 0, that of the torus is 1, while that of the surface in Figure 14 is 2. A similar surface with p holes has the genus p. The genus is a topological property of a surface and remains the same if the surface is deformed. Conversely, it may be shown (we omit the proof) that if two closed surfaces have the same genus, then one may be deformed into the other, so that the genus $p = 0, 1, 2, \ldots$ of a closed surface characterizes it completely from the topological point of view. (We are assuming that the surfaces considered are ordinary "two-sided" closed surfaces. Later in this section we shall consider "one-sided" surfaces.) For example, the two-holed doughnut and the sphere with two "handles" of Figure 15 are both closed surfaces of genus 2, and it is clear that either of these surfaces may be

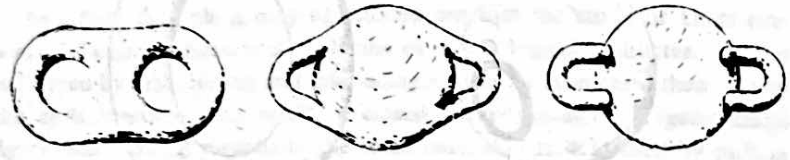

FIGURE 15—Surfaces of genus 2.

continuously deformed into the other. Since the doughnut with p holes, or its equivalent, the sphere with p handles, is of genus p, we may take either of these surfaces as the topological representative of all closed surfaces of genus p.

THE EULER CHARACTERISTIC OF A SURFACE

Suppose that a closed surface S of genus p is divided into a number of regions by marking a number of vertices on S and joining them by curved arcs. We shall show that

$$V - E + F = 2 - 2p, \tag{1}$$

where V = number of vertices, E = number of arcs, and F = number of regions. The number $2 - 2p$ is called the *Euler characteristic* of the surface. We have already seen that for the sphere, $V - E + F = 2$, which agrees with (1), since $p = 0$ for the sphere.

To prove the general formula (1), we may assume that S is a sphere with p handles. For, as we have stated, any surface of genus p may be continuously deformed into such a surface, and during this deformation the numbers $V - E + F$ and $2 - 2p$ will not change. We shall choose the deformation so as to ensure that the closed curves A_1, A_2, B_1, B_2, . . . where the handles join the sphere consist of arcs of the given subdivision. (We refer to Figure 16, which illustrates the proof for the case $p = 2$.)

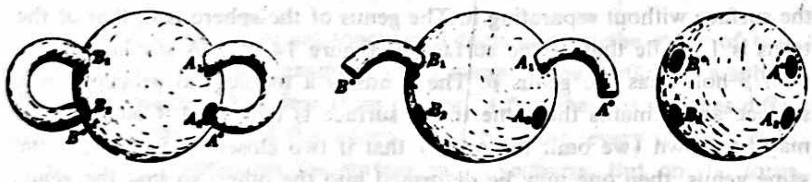

FIGURE 16

Now let us cut the surface S along the curves A_2, B_2, . . . and straighten the handles out. Each handle will have a free edge bounded by a new curve A^*, B^*, . . . with the same number of vertices and arcs as A_2, B_2, . . . respectively. Hence $V - E + F$ will not change, since the additional vertices exactly counterbalance the additional arcs, while

no new regions are created. Next, we deform the surface by flattening out the projecting handles, until the resulting surface is simply a sphere from which $2p$ regions have been removed. Since $V - E + F$ is known to equal 2 for any subdivision of the whole sphere, we have

$$V - E + F = 2 - 2p$$

for the sphere with $2p$ regions removed, and hence for the original sphere with p handles, as was to be proved.

Figure 3 illustrates the application of formula (1) to a surface S consisting of flat polygons. This surface may be continuously deformed into a torus, so that the genus p is 1 and $2 - 2p = 2 - 2 = 0$. As predicted by formula (1),

$$V - E + F = 16 - 32 + 16 = 0.$$

ONE-SIDED SURFACES

An ordinary surface has two sides. This applies both to closed surfaces like the sphere or the torus and to surfaces with boundary curves, such as the disk or a torus from which a piece has been removed. The two sides of such a surface could be painted with different colors to distinguish them. If the surface is closed, the two colors never meet. If the surface has boundary curves, the two colors meet only along these curves. A bug crawling along such a surface and prevented from crossing boundary curves, if any exist, would always remain on the same side.

Moebius made the surprising discovery that there are surfaces with only *one* side. The simplest such surface is the so-called Moebius strip, formed by taking a long rectangular strip of paper and pasting its two ends together after giving one a half-twist, as in Figure 17. A bug crawling along this surface, keeping always to the middle of the strip, will return to its original position upside down (Figure 18). Anyone who contracts to paint one side of a Moebius strip could do it just as well by dipping the whole strip into a bucket of paint.

Another curious property of the Moebius strip is that it has only one edge, for its boundary consists of a single closed curve. The ordinary two-sided surface formed by pasting together the two ends of a rectangle without twisting has two distinct boundary curves. If the latter strip is cut along the center line it falls apart into two different strips of the same kind. But if the Moebius strip is cut along this line (shown in Figure 17) we find that it remains in one piece. It is rare for anyone not familiar with the Moebius strip to predict this behavior, so contrary to one's intuition of what "should" occur. If the surface that results from cutting the Moebius strip along the middle is again cut along its middle, two separate but intertwined strips are formed.

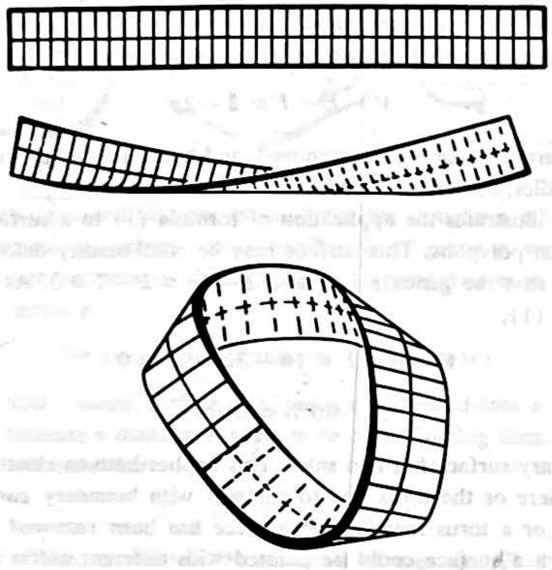

FIGURE 17—Forming a Moebius strip.

It is fascinating to play with such strips by cutting them along lines parallel to a boundary curve and ½, ⅓, etc. of the distance across. The Moebius strip certainly deserves a place in elementary geometrical instruction.

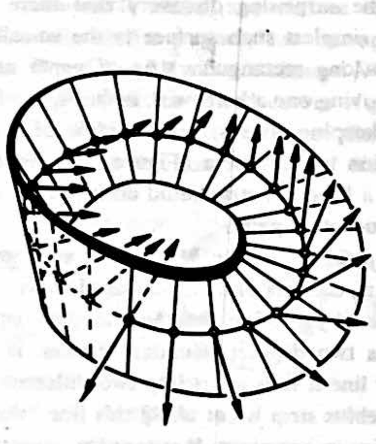

FIGURE 18—Reversal of up and down on traversing a Moebius strip.

The boundary of a Moebius strip is a simple and unknotted closed curve, and it is possible to deform it into a circle. During the deformation, however, the strip must be allowed to intersect itself. (Hence, such a

FIGURE 19—Cross-cap.

deformation of a "real" paper Moebius strip is only possible in the imagination.) The resulting self-intersecting and one-sided surface is known as a cross-cap (Figure 19). The line of intersection *RS* is regarded as two different lines, each belonging to one of the two portions of the surface which intersect there. The one-sidedness of the Moebius strip is preserved because this property is topological; a one-sided surface cannot be continuously deformed into a two-sided surface.

Another interesting one-sided surface is the "Klein bottle." This surface is closed, but it has no inside or outside. It is topologically equivalent to a pair of cross-caps with their boundaries coinciding.

FIGURE 20—Klein bottle.

It may be shown that any closed, *one-sided* surface of genus $p = 1, 2, \ldots$ is topologically equivalent to a sphere from which p disks have been removed and replaced by cross-caps. From this it easily follows that the Euler characteristic $V - E + F$ of such a surface is related to p by the equation

$$V - E + F = 2 - p.$$

The proof is analogous to that for two-sided surfaces. First we show that the Euler characteristic of a cross-cap or Moebius strip is 0. To do this we observe that, by cutting across a Moebius strip which has been subdivided into a number of regions, we obtain a rectangle that contains two more vertices, one more edge, and the same number of regions as the Moebius strip. For the rectangle, $V - E + F = 1$, as we proved on pages 581–582. Hence for the Moebius strip $V - E + F = 0$. As an exercise, the reader may complete the proof.

It is considerably simpler to study the topological nature of surfaces such as these by means of plane polygons with certain pairs of edges conceptually identified. In the diagrams of Figure 21, parallel arrows are to be brought into coincidence—actual or conceptual—in position and direction.

This method of identification may also be used to define three-dimensional closed manifolds, analogous to the two-dimensional closed surfaces. For example, if we identify corresponding points of opposite faces of a

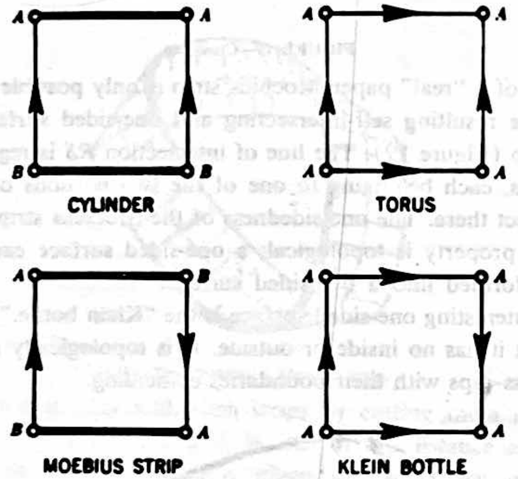

FIGURE 21—Closed surfaces defined by coördination of edges in plane figure.

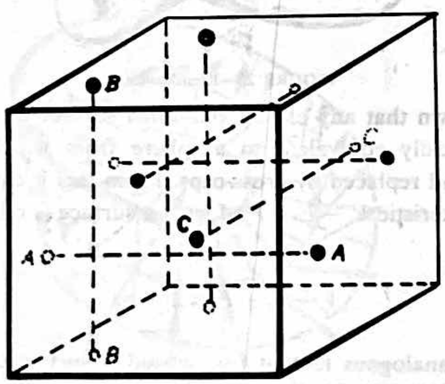

FIGURE 22—Three-dimensional torus defined by boundary identification.

cube (Figure 22), we obtain a closed, three-dimensional manifold called the three-dimensional torus. This manifold is topologically equivalent to the space between two concentric torus surfaces, one inside the other, in

which corresponding points of the two torus surfaces are identified (Figure 23). For the latter manifold is obtained from the cube if two pairs of conceptually identified faces are brought together.

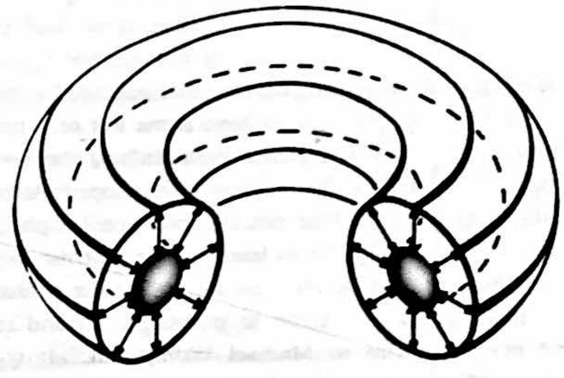

FIGURE 23—Another representation of three-dimensional torus. (Figure cut to show identification.)