garded as a small amusement of the Königsberg townsfolk. Euler, however, discovered an important scientific principle concealed in the puzzle.[4] He presented his simple and ingenious solution to the Russian Academy at St. Petersburg in 1735. His method was to replace the land areas by points and the bridges by lines connecting these points. The points are called vertices; a vertex is called odd or even according as the number of lines leading from it are odd or even. The entire configuration is a *graph*; the problem of crossing the bridges reduces to that of traversing the graph with one continuous sweep of the pencil without lifting it from the paper. Euler discovered that this can be done if the graph has only *even* vertices. If the graph contains no more than two *odd* vertices, it may be traversed in one journey but it is not possible to return to the starting point. The general principle is that if the graph contains $2n$ odd vertices, where n is any integer, it will require exactly n distinct journeys to traverse it.

Thus began a "vast and intricate theory," still young and growing, yet already one of the great forces of modern mathematics.[5]

[4] See Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*; Leipzig, 1901, second edition; vol. III, p. 552.

[5] For an entertaining and instructive account of topology, supplementary to the discussion by Courant and Robbins, see the article "Topology," by Albert S. Tucker and Herbert W. Bailey, Jr., in *Scientific American*, January 1950.

It is a pleasant surprise to him [the pure mathematician] and an added problem if he finds that the arts can use his calculations, or that the senses can verify them, much as if a composer found that the sailors could heave better when singing his songs. —GEORGE SANTAYANA

4 The Seven Bridges of Königsberg

By LEONHARD EULER

1. THE branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibnitz spoke of it first, calling it the "geometry of position" (*geometria situs*). This branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them. Recently there was announced a problem that, while it certainly seemed to belong to geometry, was nevertheless so designed that it did not call for the determination of a magnitude, nor could it be solved by quantitative calculation; consequently I did not hesitate to assign it to the geometry of position, especially since the solution required only the consideration of position, calculation being of no use. In this paper I shall give an account of the method that I discovered for solving this type of problem, which may serve as an example of the geometry of position.

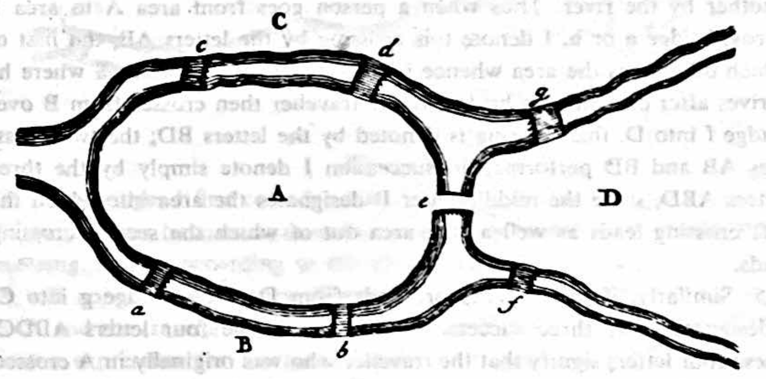

FIGURE 1

2. The problem, which I understand is quite well known, is stated as follows: In the town of Königsberg in Prussia there is an island A, called "Kneiphof," with the two branches of the river (Pregel) flowing around it, as shown in Figure 1. There are seven bridges, a, b, c, d, e, f and g, crossing the two branches. The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once. I was told that while some denied the possibility of doing this and others were in doubt, there were none who maintained that it was actually possible. On the basis of the above I formulated the following very general problem for myself: Given any configuration of the river and the branches into which it may divide, as well as any number of bridges, to determine whether or not it is possible to cross each bridge exactly once.

3. The particular problem of the seven bridges of Königsberg could be solved by carefully tabulating all possible paths, thereby ascertaining by inspection which of them, if any, met the requirement. This method of solution, however, is too tedious and too difficult because of the large number of possible combinations, and in other problems where many more bridges are involved it could not be used at all. When the analysis is undertaken in the manner just described it yields a great many details that are irrelevant to the problem; undoubtedly this is the reason the method is so onerous. Hence I discarded it and searched for another more restricted in its scope; namely, a method which would show only whether a journey satisfying the prescribed condition could in the first instance be discovered; such an approach, I believed, would be much simpler.

4. My entire method rests on the appropriate and convenient way in which I denote the crossing of bridges, in that I use capital letters, A, B, C, D, to designate the various land areas that are separated from one another by the river. Thus when a person goes from area A to area B across bridge a or b, I denote this crossing by the letters AB, the first of which designates the area whence he came, the second the area where he arrives after crossing the bridge. If the traveller then crosses from B over bridge f into D, this crossing is denoted by the letters BD; the two crossings AB and BD performed in succession I denote simply by the three letters ABD, since the middle letter B designates the area into which the first crossing leads as well as the area out of which the second crossing leads.

5. Similarly, if the traveller proceeds from D across bridge g into C, I designate these three successive crossings by the four letters ABDC. These four letters signify that the traveller who was originally in A crossed over into B, then to D, and finally to C; and since these areas are separated from one another by the river the traveller must necessarily have

crossed three bridges. The crossing of four bridges will be represented by five letters, and if the traveller crosses an arbitrary number of bridges his journey will be described by a number of letters that is one greater than the number of bridges. For example, eight letters are needed to denote the crossing of seven bridges.

6. With this method I pay no attention to which bridges are used; that is to say, if the crossing from one area to another can be made by way of several bridges it makes no difference which one is used, so long as it leads to the desired area. Thus if a route could be laid out over the seven Königsberg bridges so that each bridge were crossed once and only once, we would be able to describe this route by using eight letters, and in this series of letters the combination AB (or BA) would have to occur twice, since there are two bridges a and b, connecting the regions A and B; similarly the combination AC would occur twice, while the combinations AD, BD, and CD would each occur once.

7. Our question is now reduced to whether from the four letters A, B, C, and D a series of eight letters can be formed in which all the combinations just mentioned occur the required number of times. Before making the effort, however, of trying to find such an arrangement we do well to consider whether its existence is even theoretically possible or not. For if it could be shown that such an arrangement is in fact impossible, then the effort expended on finding it would be wasted. Therefore I have sought for a rule that would determine without difficulty as regards this and all similar questions, whether the required arrangement of letters is feasible.

8. For the purpose of finding such a rule I take a single region A into which an arbitrary number of bridges, a, b, c, d, etc., leads (Figure 2).

FIGURE 2

Of these bridges I first consider only a. If the traveller crosses this bridge he must either have been in A before crossing or have reached A after crossing, so that according to the above method of denotation the letter A will appear exactly once. If there are three bridges, a, b, c, leading to A and the traveller crosses all three, then the letter A will occur twice in the expression for his route, whether it begins at A or not. And if there are five bridges leading to A the expression for a route that crosses them all will contain the letter A three times. If the number of bridges is odd, increase it by one, and take half the sum; the quotient represents the number of times the letter A appears.

9. Let us now return to the Königsberg problem (Figure 1). Since there are five bridges, a, b, c, d, e, leading to (and from) island A, the letter A must occur three times in the expression describing the route. The letter B must occur twice, since three bridges lead to B; similarly D and C must each occur twice. That is to say, the series of eight letters that represents the crossing of the seven bridges must contain A three times and B, C and D each twice; but this is quite impossible with a series of eight letters. Thus it is apparent that a crossing of the seven bridges of Königsberg in the manner required cannot be effected.

10. Using this method we are always able, whenever the number of bridges leading to a particular region is odd, to determine whether it is possible, in a journey, to cross each bridge exactly once. Such a route exists if the number of bridges plus one is equal to the sum of the numbers that indicate how often each individual letter must occur. On the other hand, if this sum is greater than the number of bridges plus one, as it is in our example, then the desired route cannot be constructed. The rule that I gave (section 8) for determining from the number of bridges that lead to A how often the letter A will occur in the route description is independent of whether these bridges all come from a single region B, as in Figure 2, or from several regions, because I am considering only the region A, and attempting to determine how often the letter A must occur.

11. When the number of bridges leading to A is even, we must take into account whether the route begins in A or not. For example, if there are two bridges that lead to A and the route starts from A, then the letter A will occur twice, once to indicate the departure from A by one of the bridges and a second time to indicate the return to A by the other bridge. However, if the traveller starts his journey in another region, the letter A will occur only once, since by my method of description the single occurrence of A indicates an entrance into as well as a departure from A.

12. Suppose, as in our case, there are four bridges leading into the region A, and the route is to begin at A. The letter A will then occur three times in the expression for the whole route, while if the journey had started in another region, A would occur only twice. With six bridges leading to A the letter A will occur four times if A is the starting point, otherwise only three times. In general, if the number of bridges is even, the number of occurrences of the letter A, when the starting region is not A, will be half the number of the bridges; one more than half, when the route starts from A.

13. Every route must, of course, start in some one region, thus from the number of bridges that lead to each region I determine the number

of times that the corresponding letter will occur in the expression for the entire route as follows: When the number of the bridges is odd I increase it by one and divide by two; when the number is even I simply divide it by two. Then if the sum of the resulting numbers is equal to the actual number of bridges plus one, the journey can be accomplished, though it must start in a region approached by an odd number of bridges. But if the sum is one less than the number of bridges plus one, the journey is feasible if its starting point is a region approached by an even number of bridges, for in that case the sum is again increased by one.

14. My procedure for determining whether in any given system of rivers and bridges it is possible to cross each bridge exactly once is as follows: 1. First I designate the individual regions separated from one another by the water as A, B, C, etc. 2. I take the total number of bridges, increase it by one, and write the resulting number uppermost. 3. Under this number I write the letters A, B, C, etc., and opposite each of these I note the number of bridges that lead to that particular region. 4. I place an asterisk next the letters that have even numbers opposite them. 5. Opposite each even number I write the half of that number and opposite each odd number I write half of the sum formed by that number plus one. 6. I add up the last column of numbers. If the sum is one less than, or equal to the number written at the top, I conclude that the required journey can be made. But it must be noted that when the sum is one less than the number at the top, the route must start from a region marked with an asterisk. And in the other case, when these two numbers are equal, it must start from a region that does not have an asterisk.

For the Königsberg problem I would set up the tabulation as follows:

Number of bridges 7, giving 8 $(= 7 + 1)$ bridges

A,	5	3
B,	3	2
C,	3	2
D,	3	2

The last column now adds up to more than 8, and hence the required journey cannot be made.

15. Let us take an example of two islands, with four rivers forming the surrounding water, as shown in Figure 3. Fifteen bridges, marked a, b, c, d, etc., cross the water around the islands and the adjoining rivers; the question is whether a journey can be arranged that will pass over all the bridges, but not over any of them more than once. 1. I begin by marking all the regions that are separated from one another by the water with the letters A, B, C, D, E, F—there are six of them. 2. I take the number of bridges—15—add one and write this number—16—uppermost.
3. I write the letters A, B, C, etc. in a column and opposite each letter I

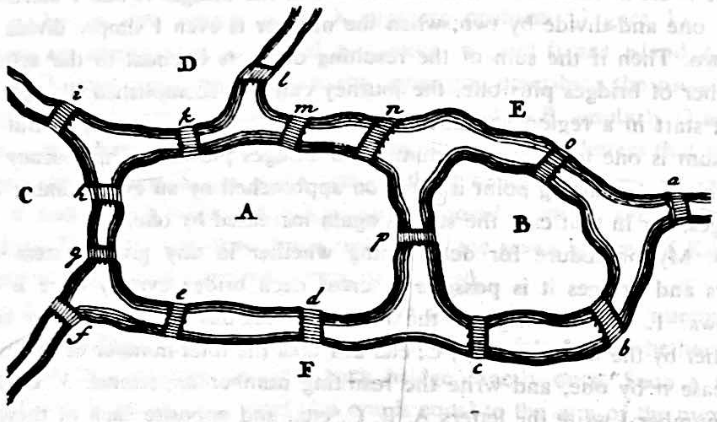

FIGURE 3

		16
A*,	8	4
B*,	4	2
C*,	4	2
D,	3	2
E,	5	3
F*,	6	3
		16

write the number of bridges connecting with that region, e.g., 8 bridges for A, 4 for B, etc. 4. The letters that have even numbers opposite them I mark with an asterisk. 5. In a third column I write the half of each corresponding even number, or, if the number is odd, I add one to it, and put down half the sum. 6. Finally I add the numbers in the third column and get 16 as the sum. This is the same as the number 16 that appears above, and hence it follows that the journey can be effected if it begins in regions D or E, whose symbols have no asterisk. The following expression represents such a route:

EaFbBcFdAeFfCgAhCiDkAmEnApBoElD.

Here I have also indicated, by small letters between the capitals, which bridges are crossed.

16. By this method we can easily determine, even in cases of considerable complexity, whether a single crossing of each of the bridges in sequence is actually possible. But I should now like to give another and much simpler method, which follows quite easily from the preceding, after a few preliminary remarks. In the first place, I note that the sum of all the numbers of bridges to each region, that are written down in the second column opposite the letters A, B, C, etc., is necessarily double the actual number of bridges. The reason is that in the tabulation of the bridges leading to the various regions each bridge is counted twice, once for each of the two regions that it connects.

17. From this observation it follows that the sum of the numbers in the second column must be an even number, since half of it represents the actual number of bridges. Hence it is impossible for exactly one of these numbers (indicating how many bridges connect with each region) to be odd, or, for that matter, three or five, etc. In other words, if *any* of the numbers opposite the letters A, B, C, etc., are odd, an even number of them must be odd. In the Königsberg problem, for instance, all four of the numbers opposite the letters A, B, C, D were odd, as explained in section 14, while in the example just given (section 15) only two of the numbers were odd, namely those opposite D and E.

18. Since the sum of the numbers opposite A, B, C, etc., is double the number of bridges, it is clear that if this sum is increased by two and then divided by 2 the result will be the number written at the top. When all the numbers in the second column are even, and the half of each is written down in the third column, the total of this column will be one less than the number at the top. In that case it will always be possible to cross all the bridges. For in whatever region the journey begins, there will be an even number of bridges leading to it, which is the requirement. In the Königsberg problem we could, for instance, arrange matters so that each bridge is crossed twice, which is equivalent to dividing each bridge into two, whence the number of bridges leading to each region would be even.

19. Further, when only two of the numbers opposite the letters are odd, and the others even, the required route is possible provided it begins in a region approached by an odd number of bridges. We take half of each even number, and likewise half of each odd number after adding one, as our procedure requires; the sum of these halves will then be one greater than the number of bridges, and hence equal to the number written at the top.

Similarly, where four, six, or eight, etc., of the numbers in the second column are odd it is evident that the sum of the numbers in the third column will be one, two, three, etc., greater than the top number, as the case may be, and hence the desired journey is impossible.

20. Thus for any configuration that may arise the easiest way of determining whether a single crossing of all the bridges is possible is to apply the following rules:

If there are more than two regions which are approached by an odd number of bridges, no route satisfying the required conditions can be found.

If, however, there are only two regions with an odd number of approach bridges the required journey can be completed provided it originates in one of the regions.

If, finally, there is no region with an odd number of approach bridges, the required journey can be effected, no matter where it begins. These rules solve completely the problem initially proposed.

21. After we have determined that a route actually exists we are left with the question how to find it. To this end the following rule will serve: Wherever possible we mentally eliminate any two bridges that connect the same two regions; this usually reduced the number of bridges considerably. Then—and this should not be difficult—we proceed to trace the required route across the remaining bridges. The pattern of this route, once we have found it, will not be substantially affected by the restoration of the bridges which were first eliminated from consideration—as a little thought will show; therefore I do not think I need say more about finding the routes themselves.

I must go in and out. —BERNARD SHAW (*Heartbreak House*)

5 Topology

By RICHARD COURANT and HERBERT ROBBINS

EULER'S FORMULA FOR POLYHEDRA

ALTHOUGH the study of polyhedra held a central place in Greek geometry, it remained for Descartes and Euler to discover the following fact: In a simple polyhedron let V denote the number of vertices, E the number of edges, and F the number of faces; then always

$$V - E + F = 2. \quad (1)$$

By a *polyhedron* is meant a solid whose surface consists of a number of polygonal faces. In the case of the regular solids, all the polygons are congruent and all the angles at vertices are equal. A polyhedron is *simple* if there are no "holes" in it, so that its surface can be deformed continuously into the surface of a sphere. Figure 2 shows a simple polyhedron which is not regular, while Figure 3 shows a polyhedron which is not simple.

The reader should check the fact that Euler's formula holds for the simple polyhedra of Figures 1 and 2, but does not hold for the polyhedron of Figure 3.

To prove Euler's formula, let us imagine the given simple polyhedron to be hollow, with a surface made of thin rubber. Then if we cut out one of the faces of the hollow polyhedron, we can deform the remaining surface until it stretches out flat on a plane. Of course, the areas of the faces and the angles between the edges of the polyhedron will have changed in this process. But the network of vertices and edges in the plane will contain the same number of vertices and edges as did the original polyhedron, while the number of polygons will be one less than in the original polyhedron, since one face was removed. We shall now show that for the plane network, $V - E + F = 1$, so that, if the removed face is counted, the result is $V - E + F = 2$ for the original polyhedron.

First we "triangulate" the plane network in the following way: In some polygon of the network which is not already a triangle we draw a diagonal. The effect of this is to increase both E and F by 1, thus pre-