# MATHEMATICAL GAMES

## *Diversions that clarify group theory, particularly by the weaving of braids*

by Martin Gardner

The concept of "group," one of the great unifying ideas of modern algebra and an indispensable tool in physics, has been likened by James R. Newman to the grin of the Cheshire cat. The body of the cat (algebra as traditionally taught) vanishes, leaving only an abstract grin. A grin implies something amusing. Perhaps we can make group theory less mysterious if we do not take it too seriously.

Three computer programmers—Ames, Baker and Coombs—wish to decide who pays for the beer. Of course they can flip pennies, but they prefer a random decision based on the following network-tracing game. Three vertical lines are drawn on a sheet of paper. One programmer, holding the paper so that his friends cannot see what he is doing, randomly labels the lines A, B and C [*see illustration at left below*]. He folds back the top of the sheet to conceal these letters. A second player now draws a series of random horizontal lines—call them shuttles—each connecting two of the vertical lines [*see second illustration below*]. The third player adds a few more shuttles, then marks an X at the bottom of one of the vertical lines [*see third illustration*].

The paper is unfolded. Ames puts his finger on the top of line A and traces it downward. When he reaches the end of a shuttle he turns, follows the shuttle to its other end, turns again and continues downward until he reaches the end of another shuttle. He keeps doing this until he reaches the bottom. His path [*traced in color in the fourth illustration*] does not end on the X, so he does not have to buy the drinks. Baker and Coombs now trace their lines in similar fashion. Baker's path ends on the X, so he picks up the tab. For any number of vertical lines, and regardless of how the shuttles are drawn, each player will always end on a different line.

A closer look at this game discloses that it is based on one of the simplest of groups, the so-called permutation group for three symbols. What, precisely, is a group? It is an abstract structure involving a set of undefined elements ($a, b, c \ldots$) and a single undefined operation (here symbolized by . ) that pairs one element with another to produce a third. The structure is not a group unless it has the following four traits:

1. When two elements of the set are combined by the operation, the result is another element in the same set. This is called "closure."

2. The operation obeys the "associative law": $(a.b).c = a.(b.c)$

3. There is one element $e$ (called the "identity") such that $a.e = e.a = a$

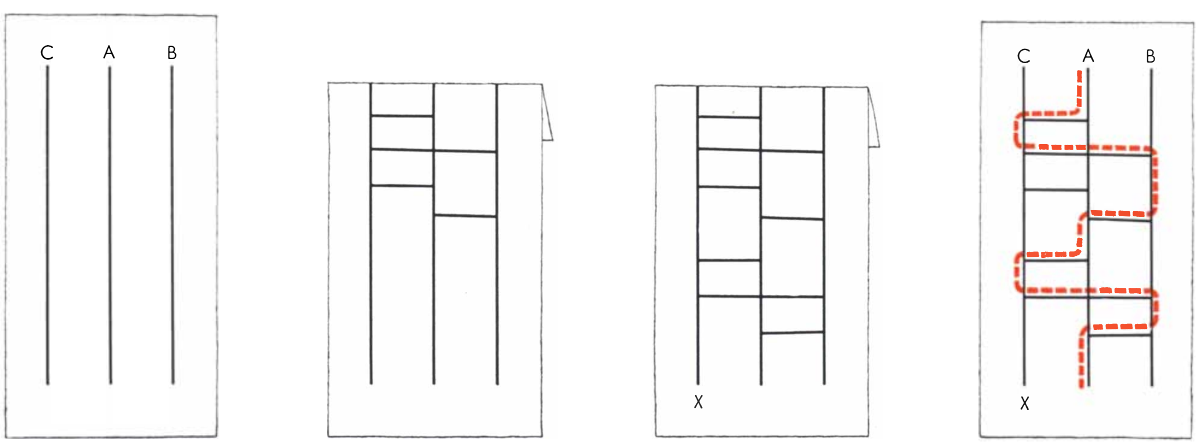
4. For every element $a$ there is an inverse element $a'$ such that $a.a' = a'.a = e$

If in addition to these four traits the operation also obeys the commutative law ($a.b = b.a$), the group is called a commutative or Abelian group.

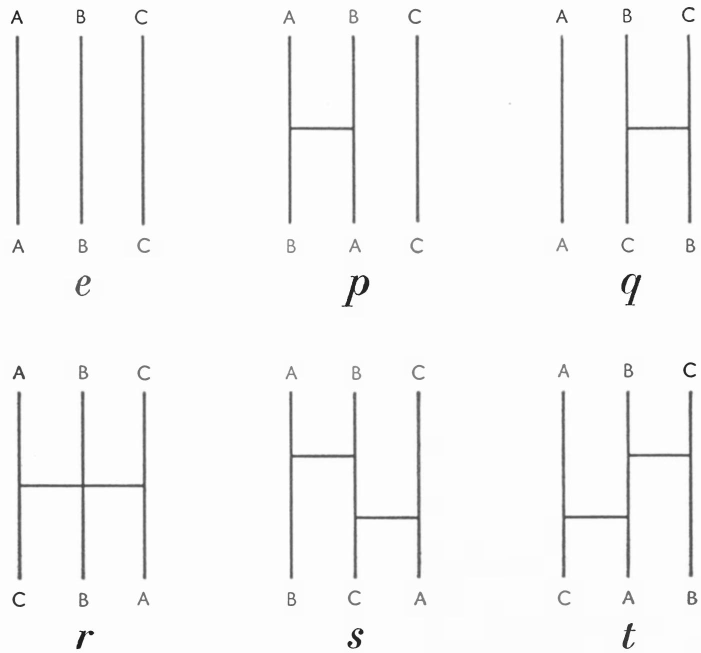
The most familiar example of a group is provided by the integers with respect to the operation of addition. It has closure (any integer plus any integer is an integer). It is associative (adding 2 to 3 and then adding 4 is the same as adding 2 to the sum of 3 and 4). The identity is 0 and the inverse of a positive integer is the negative of that integer. It is an Abelian group (2 plus 3 is the same as 3 plus 2). The integers also form an Abelian group with respect to multiplication, but here the identity is 1 and the inverse of an integer is its reciprocal (*e.g.*, the inverse of 5 is 1/5). The integers do *not* form a group with respect to division: 5 divided by 2 is 2½, which is not an element in the set.

Let us see how the network game exhibits group structure. The top illustration on page 169 depicts the six basic "transformations" that are the elements of our finite group. Transformation $p$ switches the paths of A and B so that the three paths end in the order BAC. Transformations $q$, $r$, $s$ and $t$ give other permutations. Transformation $e$ is not much of a change; it consists of drawing no shuttles at all. These six elements correspond to the six different ways in which three symbols can be permuted. Our group operation, symbolized by . , is simply that of following one transformation with another; that is, of adding shuttles.

A quick check reveals that we have




*The network-tracing game*



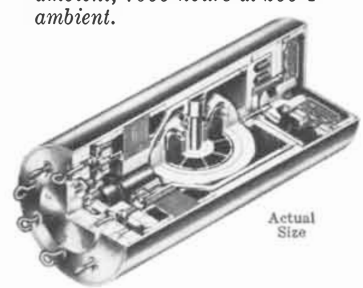

*The six elements of the network-game group*

here a structure with all the properties of a group. It has closure because no matter how we pair the elements we always get a permutation in the order of the paths that can be achieved by one element alone. For example, $p.t = r$ because $p$ followed by $t$ has exactly the same effect on the path order as applying $r$ alone. The operation of adding shuttles is clearly associative. Adding no shuttles is the identity. Elements $p$, $q$ and $r$ are their own inverses, and $s$ and $t$ are inverses of each other. (When an element and its inverse are combined,

| | *e* | *p* | *q* | *r* | *s* | *t* |
|---|---|---|---|---|---|---|
| *e* | *e* | *p* | *q* | *r* | *s* | *t* |
| *p* | *p* | *e* | *s* | *t* | *q* | *r* |
| *q* | *q* | *t* | *e* | *s* | *r* | *p* |
| *r* | *r* | *s* | *t* | *e* | *p* | *q* |
| *s* | *s* | *r* | *p* | *q* | *t* | *e* |
| *t* | *t* | *q* | *r* | *p* | *e* | *s* |

*Results of pairing elements in the network-game group. Broken line indicates* $r \, . \, s = p$



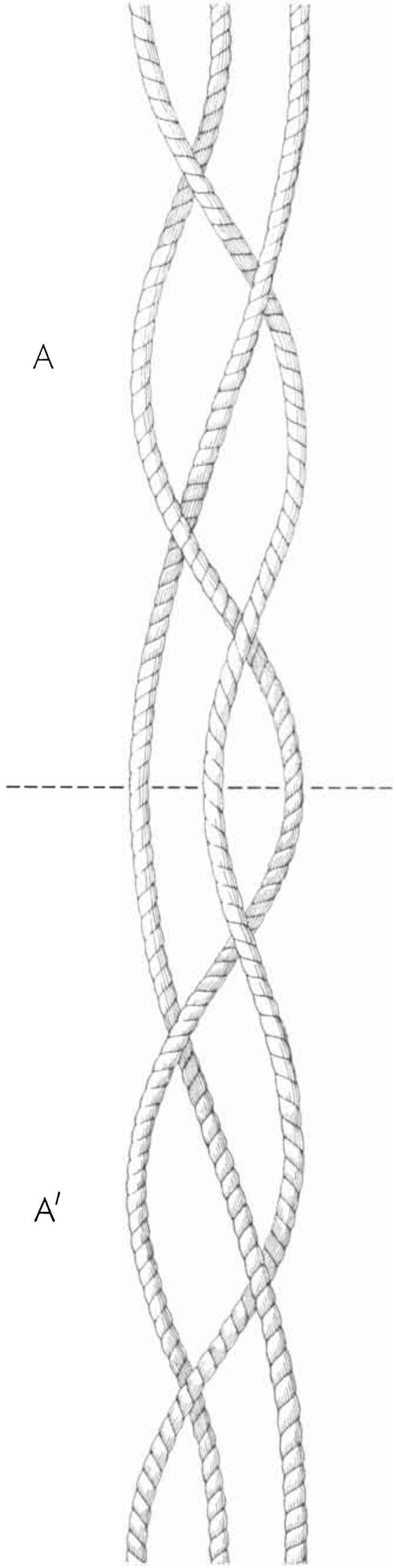


the result is the same as drawing no shuttles at all.) It is not an Abelian group (*e.g.*, *p* followed by *q* is not the same as *q* followed by *p*).

The bottom illustration on page 169 provides a complete description of this group's structure. What is the result of following *r* with *s*? We find *r* on the left side of the table and *s* at the top. The intersection of column and row is the cell labeled *p*. In other words, shuttle pattern *r* followed by shuttle pattern *s* has the same effect on path order as pattern *p*. This is a very elementary group that turns up in many places. For example, if we label the corners of an equilateral triangle, then rotate and reflect the triangle so that it always occupies the same position on the plane, we find that there are only six basic transformations possible. These transformations have the same structure as the group just described.

It is not necessary to go into group theory to see intuitively that the network game will never permit two players to end their paths on the same vertical line. Simply think of the three lines as three ropes. Each shuttle has the same effect on path order as crossing two ropes, as though forming a braid. Obviously no matter how you make the braid or how long it is, there will always be three separate lower ends.

Let us imagine that we are braiding three strands of a girl's hair. We can record successive permutations of strands by means of the network diagram, but it will not show how the strands pass over and under one another. If we take into account this complicating topological factor, is it still possible to call on group theory for a description of what we are doing? The answer is yes, and Emil Artin, a distinguished mathematician now at the University of Hamburg, was the first to prove it. In his elegant theory of braids the elements of the group are "weaving patterns" (infinite in number), and the operation consists, as in the network game, of following one pattern with another. As before, the identity element is a pattern of straight strands—the result of doing nothing. The inverse of a weaving pattern is its mirror image. The illustration at right shows a sample pattern followed by its inverse. Group theory tells us that when an element is added to its inverse, the result is the identity. Sure enough, the two weaving patterns combined prove to be topologically equivalent to the identity. A tug on the end of the braid in the illustration and all strands pull out straight. (Many magic tricks with rope, known




*Braid A is the mirror image of A′*

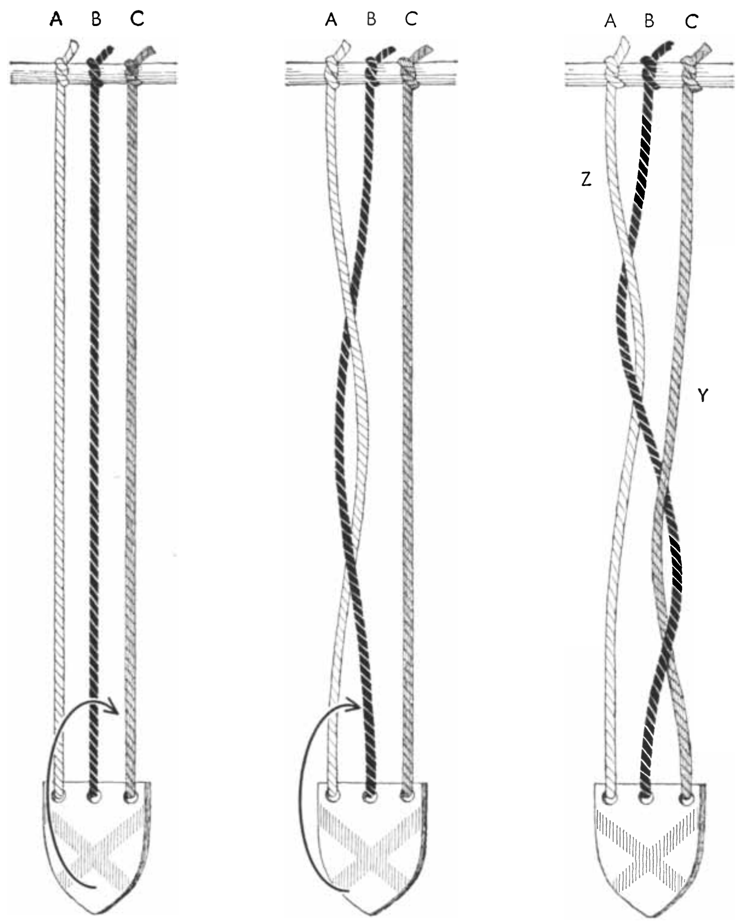
in the trade as releases, are based on this interesting property of groups.) Artin's theory of braids not only provided for the first time a system that classified all types of braids; it also furnished a method by which one could determine whether two weaving patterns, no matter how complex, were or were not topologically equivalent.

Braid theory is involved in an unusual game devised by Piet Hein of Copenhagen, several of whose mathematical recreations have been discussed in this department. Cut a piece of heavy cardboard into the coat-of-arms shape depicted below. This will be called the plaque. Its two sides must be easily distinguished, so color one side or mark it with an X as shown. Punch three holes at the square end. A two-foot length of heavy but flexible cord (sash cord is excellent) is knotted to each hole. The other ends of the three strands are attached to some fixed object like the back of a chair.

You will find that the plaque can be given complete rotations in six different ways to form six different braids. It can be rotated sidewise to the right or to the left; it can be rotated forward or backward between strands A and B; it can be rotated forward or backward between strands B and C. The second illustration below shows the braid obtained by a forward rotation through B and C. The question arises: Is it possible to untangle this braid by weaving the plaque in and out through the strands, keeping it horizontal at all times, X-side up, and always pointing toward you? The answer is no. But if you give the plaque a second rotation, in any of the six different ways, the result is a braid that *can* be untangled by weaving the plaque without rotating it.

To make this clear, assume that the second rotation is forward between A and B, creating the braid shown in the third illustration. To remove this braid without rotating the plaque, first raise C at the spot marked Y and pass the plaque under it from right to left. Pull the strings taut. Next raise A at the spot marked Z and pass the plaque under it from left to right. The result is that the cords pull straight.

The following surprising theorem

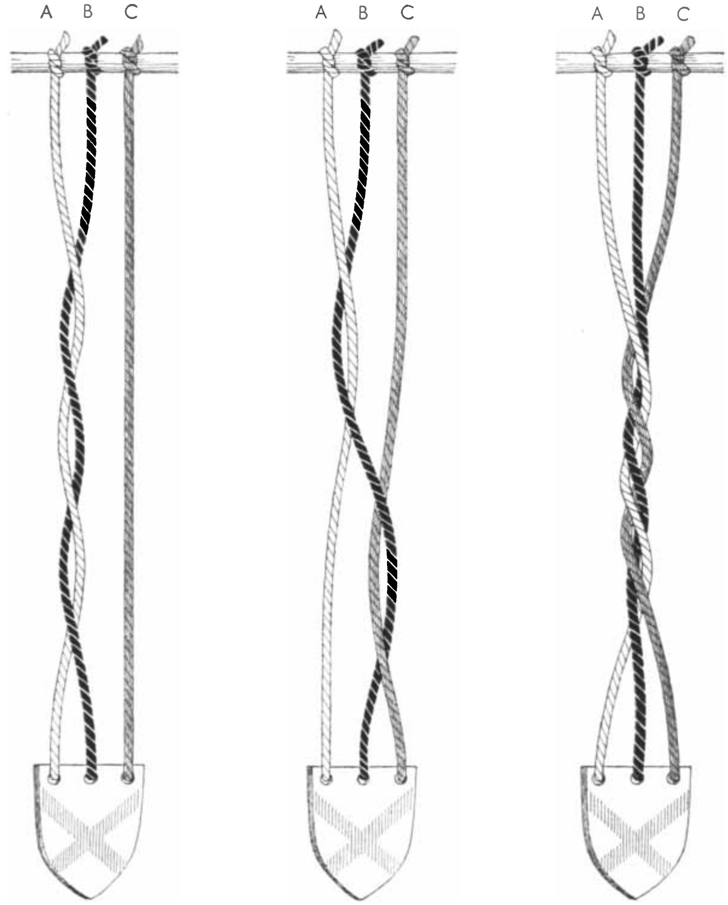



*Rotation at left produces braid in center; rotation in center, braid at right* holds for any number of strands above two. All braids produced by an *even* number of rotations (each rotation may be in any direction whatever) can always be untangled by weaving the plaque without rotating it; braids produced by an *odd* number of full rotations can never be untangled.

It was at a meeting in Niels Bohr's Institute for Theoretical Physics, more than 25 years ago, that Hein first heard this theorem discussed by Paul Ehrenfest in connection with a problem in quantum theory. A demonstration was worked out, by Hein and others, in which Mrs. Bohr's scissors were fastened to the back of a chair with strands of cord. It later occurred to Hein that the rotating body and the surrounding universe entered symmetrically into the problem and therefore that a symmetrical model could be created simply by attaching a plaque to *both* ends of the cord. With this model two persons can play a topological game. Each holds a plaque, and the three strands are stretched straight between the two plaques. The players take turns, one forming a braid and the other untangling it, timing the operation to see how long it takes. The player who untangles the fastest is the winner.

The odd-even theorem also applies to this two-person game. Beginners should limit themselves to two-rotation braids, then proceed to higher even-order braids as they develop skill. Hein calls his game "tangloids," and it has been played in Europe for a number of years.

Why do odd and even rotations make such a difference? This is a puzzling question that is difficult to answer without going more deeply into group theory. A hint is supplied by the fact that two rotations in exactly opposite directions naturally amount to no rotation. And if two rotations are almost opposite, prevented from being so only by the way certain strings pass around the plaque, then the tangle can be untangled by moving those same strings back around the plaque. M. H. A. Newman, in an article published in a London mathematical journal in 1942, says that P. A. M. Dirac, the noted University of Cam-




*Three problems of braid disentanglement*

*Solution to last month's card problem*

bridge physicist, has for many years used the solitaire form of this game as a model "to illustrate the fact that the fundamental group of the group of rotations in 3-space has a single generator of the period 2." Newman then draws on Artin's braid theory to prove that the cords cannot be untangled when the number of rotations is odd.

You will find it a fascinating pastime to form braids by randomly rotating the plaque an even number of times, then seeing how quickly you can untangle the cords. Three simple braids, each formed by two rotations, are shown in the illustration on page 174. The braid on the left is formed by rotating the plaque forward twice through B and C; the braid in center, by rotating the plaque forward through B and C and then backward through A and B; the braid at right, by two sidewise rotations to the right. Readers are invited to determine the best method of untangling each braid. Answers will be given in the next issue.

Following a practice inaugurated last December, I close with a cryptic Yuletide message. To find it, you must permute properly all the letters (that is, form an anagram) of the following sentence: MANY A SAD HEART CAN WHISPER MY PRAYER.

Unfortunately permutation group theory is of no help here, but the puzzle is not nearly so hard as it looks.

Last month readers were asked to form a square with the 16 highest playing cards so that no value or suit would appear twice in any row, column or two main diagonals. One solution is given in the illustration above.