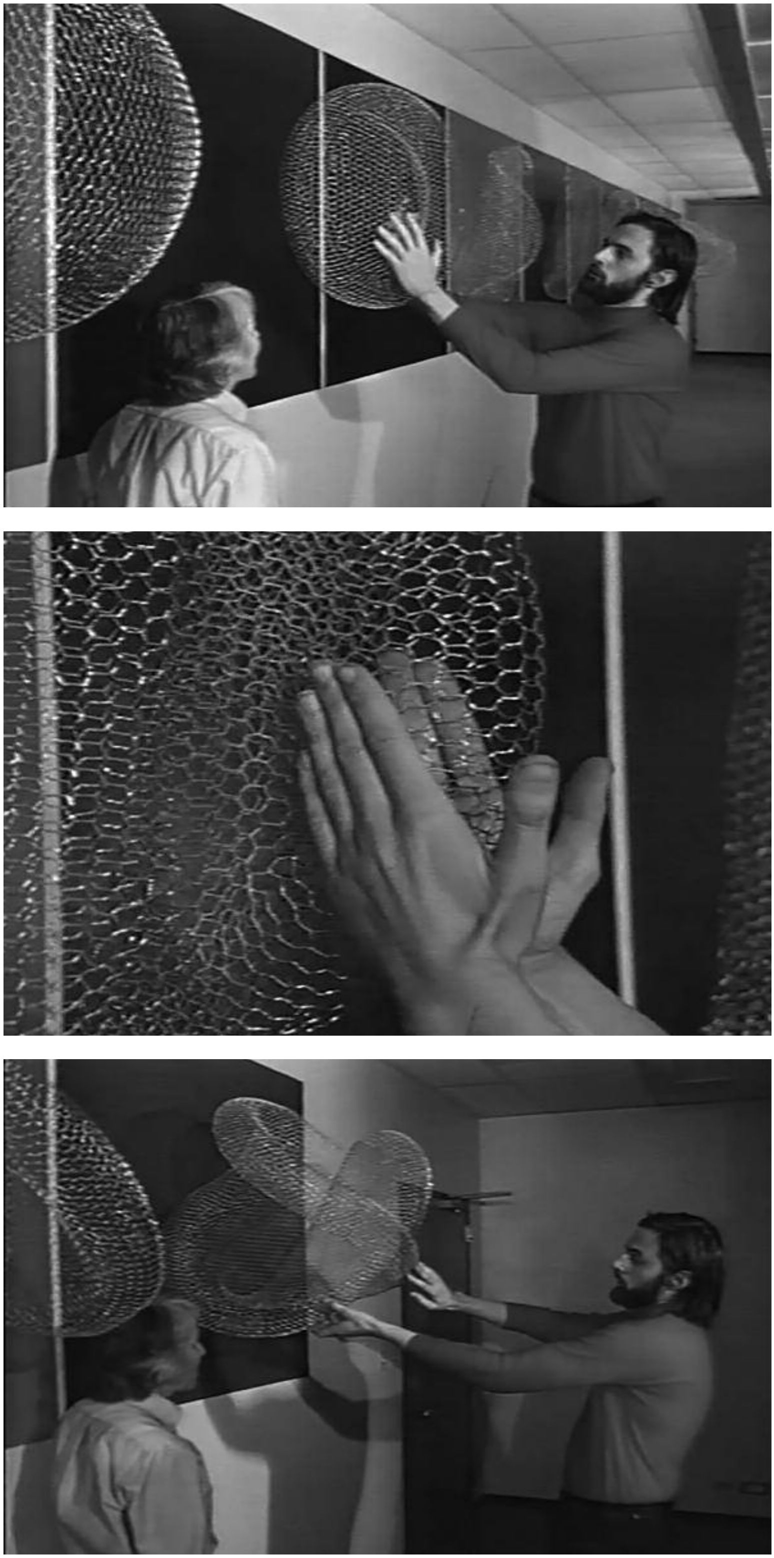

Turning a Sphere Inside Out, dir. Nelson Max, 1977. Charles Pugh's chicken-wire models of sphere eversion.

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Inside: Out

ALMA STEINGART

> *The word "image" is in bad repute because we have thoughtlessly believed that a design was a tracing, a copy, a second thing, and that the mental image was such a design, belonging among our private bric-a-brac. But if in fact it is nothing of the kind, then neither the design nor the painting belongs to the in-itself any more than the image does. They are the inside of the outside and the outside of the inside, which the duplicity of feeling makes possible and without which we would never understand the quasi presence and imminent visibility which makes up the whole problem of the imaginary.*
> —Maurice Merleau-Ponty[1]

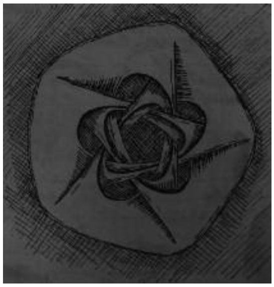
"The reason I was doing it was because I was trying to understand the geometry that's implied by Steve's theorem. Steve's theorem is completely abstract and does not mention too much geometry in it. At least you can't *see* the geometry. These models are an attempt to *see* what's going on."[2] Sporting a crimson turtleneck, shoulder-length hair, and verdant beard, mathematician Charles Pugh gesticulates as he explains to physicist Judith Bergman what inspired him to spend hours of his free time constructing 1.6-foot chicken-wire models of a solution to a topological problem known as the sphere eversion: turning a sphere inside out.[3] Both Pugh's sartorial choices and the 16 mm film's desaturated colors and tinny tones mark it as a product of the mid-1970s. Topological models hang from the ceiling behind Pugh, suspended from the walls of the newly opened Evans Hall, home to the department of mathematics at the University of California, Berkeley. Running the length of the wall, a succession of green, blue, and brown squares mounted behind the models ensures that the sculpted forms are clearly visible and do not recede into the wall behind them. Sitting on a computer chair to Pugh's right, Stephen Smale (who proved the theorem in 1957) interjects into the conversation. The camera moves to the left as Smale smiles and adds, "I take exception to you saying that my theory is completely abstract."[4] The two men laugh, as Pugh partially concedes, "yes, in principle, you can make geometric models from your theory. However," he is quick to add, "no one was able to."[5] The friendly squabble is quickly left behind, as Pugh moves on to explain how he built his models.

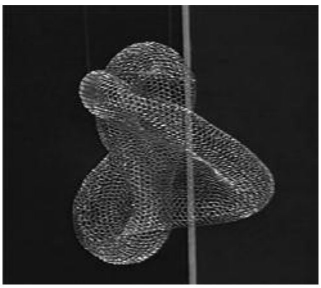
In Smale and Pugh's brief interaction, I detect two modes of mathematical engagement, which can broadly be described as an abstract and generalized approach on the one hand and a concrete and perceptual one on the other. One of the clearest examples of how mathematicians in the second half of the twentieth century have actively sought to merge the two and move beyond mere symbolic reasoning is the history of the sphere eversion, which has challenged mathematicians' apprehension and perception for the past five decades. By following this story, I examine how these mathematicians traffic between the theoretical and the tangible, an idea and its instantiation, the abstract and the concrete.

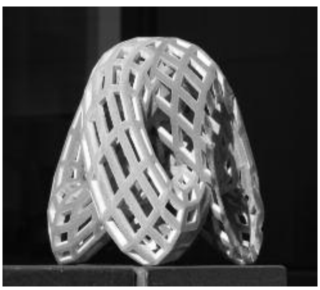
The models Pugh displayed at Berkeley were neither the first nor the last solution mathematicians have devised to the sphere eversion problem. Like a classic novel adapted over and over again for screen and stage, during the past half-century mathematicians have constructed several solutions to the problem and have done so in multiple media. They have drawn the eversion on paper and modeled it in clay, chicken wire, plaster, fiberglass, and even ice. And they have programmed the eversion time and time again on computer screens.

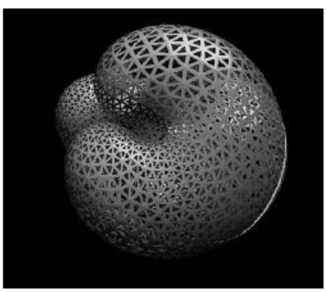
What inspired mathematicians repeatedly to return to the same topological problem? Those who worked on the problem have overwhelmingly noted that their main motivation was a desire *to see* a sphere eversion. But despite their reliance on material and virtual models, their desire *to see* was never restricted to their wish to perceive the deformation with their *eyes*. As Pugh explained, his models provided means with which to "*understand* the geometry" implied in the theorem. To see, to understand, and to visualize were all used interchangeably by the mathematicians whose work I detail here. For them, seeing was never restricted to sensorial perception, visualizing was not confined to the sense of sight, and understanding was always enmeshed within a multitude of bodily perceptions.[6] Seeing entailed a theoretical comprehension, and visualizing implied embodied apprehension. For the mathematicians who over the decades worked on this problem, the cognitive and sensorial were constituted in tandem.

Material practices have played an important (albeit too often neglected) role in the long history of mathematics—especially in geometry. From early Greek diagrams of Euclidean proofs, to geometrical treatises of the Middle Ages, to early modern mechanical instruments, to nineteenth-century three-dimensional string and plaster models of geometrical surfaces, historians have noted the prevalence of material artifacts in the production and dissemination of mathematical knowledge.[7] However, with the rise of mathematical modernism at the end of the nineteenth century, visualization techniques were for the most part banished from mathematics.[8] The rise of non-Euclidean and higher-dimensional geometry in the









Top to bottom, left to right:
Bernard Morin. Plaster model of sphere eversion, ca. 1960s. Courtesy Anthony Phillips.

"Turning a Snowball Inside Out," from the Fourteenth International Snow Sculpting Championships, 2004. Courtesy Carlo H. Séquin.

Turning a Sphere Inside Out, dir. Nelson Max, 1977.

Alex Kozlowski and Carlo H. Séquin. Model of sphere eversion created with 3-D printer, 2004. Courtesy Carlo H. Sequin.

Cover illustration, *Scientific American*, 1966.

George Francis. The tobacco pouch eversion, 1980.

Jim Blinn. Computer rendering for *Turning a Sphere Inside Out*, 1976.

A computer rendering produced as part of the work on *Outside In,* 1994. © The Geometry Center.

John Hughes. A computer implementation of Bernard Morin's eversion, 1989.

The Optiverse, dir. John M. Sullivan, George Francis, and Stuart Levy, 1998.

nineteenth century transformed geometry: no longer merely the study of the lived physical world, geometry instead became the study of abstract space. And not just geometry—by the end of the century mathematics more broadly began to be conceived of as the study of arbitrary axioms bearing no necessary relation to the world.[9]

Consequently, mathematics bifurcated into pure and applied domains.[10] The emergence of pure mathematics as an independent field of study was perfectly complemented by the rise, at the turn of the century, of structural objectivity, which, according to Lorraine Daston and Peter Galison, eschewed all "images, whether they are perceived by the eye of the body or that of the mind."[11] A formalistic and axiomatic approach increasingly dominated mathematics, to the extent that soon mathematics was widely defined as the science of structures.[12] This transformation, which espoused generalization and unification above all, accounts in part for the diminishing role of illustrations and models in mathematical practice.[13] According to historian of mathematics Herbert Mehrtens,

> the neglect of 3-D models (and 2-D diagrams) during most of the twentieth century is closely related to the dominance of mathematical modernism with its preference for general theory, symbolic formalism, and the treatment of mathematical theories as worlds of their own without an immediate relation to the physical world around us.[14]

In lieu of the perceptual world, abstraction reigned.[15]

For the American mathematical community, this trend toward abstraction and generalization was especially pronounced. Unlike its European counterpart it did not have a long tradition of research in applied mathematics. Even after World War II, and despite bitter internal fights, most American mathematicians continued to promote the study of mathematics as independent of the physical world.[16] Defined in opposition to applied mathematics, pure mathematical research maintained its autonomy by continuing to espouse a generalized and theoretical approach to mathematical research. The "essential condition" of the growth of mathematics during the twentieth century, Marshall Stone proclaimed in 1956, was that mathematicians "recognized and acted upon the fact that mathematics is not closely bound to the material world or to the physical reality—if, indeed, it is bound at all."[17] The future of mathematics could be found not only in "technical virtuosity" but in "abstraction and universality."[18] As such, twentieth-century mathematics departed from the particular, the concrete, and the sensually apprehensible.

In the exchange between Pugh and Smale, one can detect a noticeable undercurrent, a growing trend among mathematicians in the 1970s to resurrect and reestablish the perceptual aspect of mathematical practice.[19] This was done not in the name of objectivity but to reclaim additional ways of mathematical sense-making that extended beyond formalism.[20] When Pugh explained to Bergman that "Steve's theorem is completely abstract," he did not question the veracity of the work, its claim to mathematical certitude; rather, he sought to find a more tangible articulation of it. His models, and the various computer animations that followed them, were intended not as a replacement but as a supplement, a means of achieving a more immediate apprehension.

The history of sphere eversion, therefore, is a particular example of a widespread mode of mathematical practice that I term *mathematical manifestation*. I use this term to mark the concrete, demonstrable, exemplary, displayable, and presentable aspects of mathematical research.[21] Beyond their use in pedagogical work or as an aid to theorem-proving, manifestations are generative and denote the way mathematicians use material explorations to approach and generate embodied understanding of otherwise abstract principles.[22] I use *manifestation* rather than the more commonly used *visualization* to signal that seeing is only one of several ways in which mathematicians work to attain an embodied, experiential understanding of mathematical concepts and objects.[23]

The story I here detail tracks and in some ways recapitulates in miniature a history of animation. In their efforts to model the sphere eversion, topologists progressed from a zootrope approach, to stop-motion or Claymation-style animation, to computer graphics interpolation, and finally to fully programmable three-dimensional computer animation.[24] Theoretical developments in topology and advancements in computer graphic technology propelled their attempts. However, tracking these successive attempts, I focus less on how each new iteration amended and improved the one that came before it and more on how each mediated approach both assumed and cultivated a specific mode of mathematical apprehension. The first attempts to represent the deformation through line-drawn illustrations used optic projection to place mathematical observers at a distance from the object of their investigation. These drawings required that mathematicians imaginatively meld each discrete drawing to the next to produce the effect of seamless movement. The gap between mathematician and topological object diminished when mathematicians next began building physical models of the eversion. Their imaginations still labored to embed discrete models in an animated topological process, but instead of projective vision, tactile and kinesthetic sensation evinced this transformation. The third attempt to model the eversion, using computer graphics to interpolate from one static image to the next, merged the optic and the tactile in a mode of haptic visuality.[25] Finally, vir-

Netta Steingart. The inadmissible solution to sphere eversion, 2013.

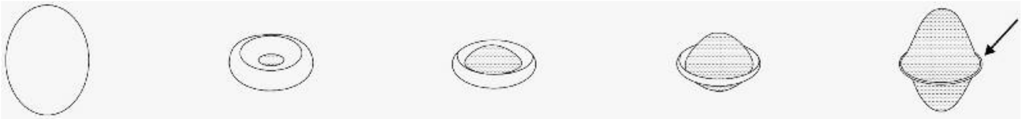
tual reality placed the mathematician inside an immersive and agential environment. Here, movement does not appear in front of but rather *engulfs* the observer.

The Phenomenological Sphere

The sphere eversion problem belongs to the field of topology, which can be loosely defined as the field that studies the properties of geometric objects that are preserved under continuous deformation. The field is relatively young. Although some topological notions can be traced back to the eighteenth century, the origin of the field is most often dated to the work of Henri Poincaré at the end of the nineteenth century. From a pure disciplinary perspective, the field began coalescing during the first two decades of the twentieth century, when mathematicians began to expand Poincaré's ideas.[26] As such, topology, with its emphasis on generalizations and abstractions, can safely be classified as one of the modern subfields of mathematics. As historian of mathematics Jeremy Gray notes, despite the fact that "topology can be given almost a kindergarten feel at times, [it] is actually a highly abstract subject in which the objects and methods are often inaccessible without axioms."[27] Despite the seemingly intuitive conception of the field as dealing with the deformation of objects in space, the study of topology advanced by considering the algebraic properties of these surfaces.[28]

Topology is often described as "rubber-sheet-geometry" since surfaces can be "twisted" and "stretched" but cannot be "torn" or "punctured." Thus, from a topological (as opposed to a geometrical) point of view, a circle and an ellipse are equivalent because one can be deformed to the other. Stated more accurately, one can construct a one-to-one map between points on the circle and those on the ellipse, a map that translates nearby points to nearby points. The sphere eversion problem poses a deceptively simple question: Can you take a sphere and turn it inside out without tearing, puncturing, or creating creases in its surface? Mathematically, it asks whether there exists a *regular homotopy* between the sphere and its eversion.[29] A differential function from the standard round spherical surface (S^2) to three-dimensional Euclidean space (R^3) is called an *immersion* of the sphere if its derivative taken at any given point on the sphere is injective. A regular homotopy is a family of continuous immersions whose partial derivatives with respect to position and time are continuous. The notion of a homotopy, which was first introduced by Poincaré, is useful for topologists as an invariant that enables them to classify surfaces up to a given equivalence.[30]

Over the years mathematicians have described the problem in several popular accounts. In translating formal mathematics into plain language, these descriptions are remarkably uniform: "Imagine that the surface of the sphere is made out



of rubber like a hollow ball," begins one such rendition. "If we make a hole in the ball we can turn it inside out through the hole." However, the author quickly adds that such a move would have "destroyed the surface."[31] Thus, in order to explain the intricate complexity of the problem, the first thing to establish is which deformations are permissible and which are not. Tearing the surface is not allowed, but letting two surfaces pass through one another is acceptable. "A physical sheet of rubber can never cross itself, but a mathematical surface in space can."[32] The definition of immersion given above is not a one-to-one function. That is, two different points of S^2 can be mapped to the same point in R^3. Throughout the deformation, a single point in space can be the meeting point of two or more planes.

If no further restrictions existed, then the problem would not be too difficult to solve; as one account declares, "a solution suggests itself."[33] The inadmissible solution, which is repeated in every portrayal, involves pushing two opposing regions of the sphere (the "north pole" and the "south pole") through one another. However, topologists are quick to point out that as the two poles move through one another a ridge slowly forms across the equator that eventually results in a crease just before the deformation is complete. The crease is "disagreeable" and "displeasing to differential topologists," because the surface is no longer smooth.[34] Smoothness is the requirement that the partial derivatives are continuous in relation to time and position.

At one time most mathematicians believed a sphere could not be everted.[35] Yet, in 1957 Smale surprised the mathematical community when he published a paper in the *Transactions of the American Mathematical Society* formally proving that a sphere could be turned inside out following the rules of topology.[36] If theorem-proving was the sole goal of mathematicians, then Smale's publication should have brought an end to their interest in the problem. After all, Smale showed once and for all that a regular homotopy existed. Yet just the opposite happened. Smale's proof demonstrated that a way to turn a sphere inside out *must* exist, but, as Pugh noted, it did not indicate *how* to do so. Mathematicians who read Smale's paper and followed its proof knew a sphere theoretically could be turned inside out, but they did not know *how* or *why*. What Smale had proved was a broader and more robust theorem concerning the classification of immersed spheres. According to Smale, when he realized that the theorem he was trying to prove implied a sphere could be turned inside out, he tried to find an explicit solution. His numerous attempts were futile, but he became convinced that a solution, contrary to common belief, must exist. Smale noted that these attempts gave him the "confidence to go ahead and complete the analytic proof required by the general theory."[37]

Top: René Thom. Letter to Anthony Phillips, 1964.

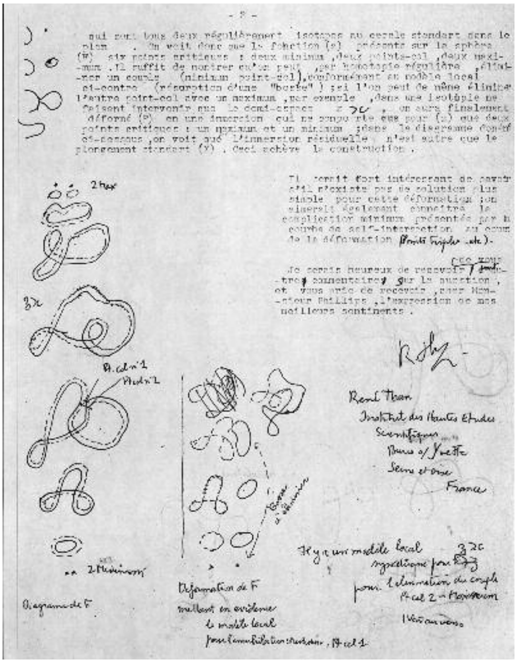
Bottom: Anthony Philips. Sketches, ca. 1966.

Nonetheless, most mathematicians who read Smale's paper were left wanting a direct demonstration.[38] Thus, instead of putting the problem to rest, the publication of Smale's paper only ignited further interest in the problem as mathematicians turned their attention to finding explicit solutions. In trying to manifest these solutions, the main problem mathematicians faced was how to depict a continuous transformation in three dimensions. In the 1960s, mathematicians worked with drawings and sculpted three-dimensional models. Beginning in the 1970s, most efforts coalesced around the rise of computer graphics, making the problem emblematic of the growing use of computer visualization in mathematics. As some mathematicians and computer graphics specialists described the problem, "If a rendered teapot is the classical subject of computer graphics, sphere eversion is the 'teapot' of visualizable geometry."[39] Nonetheless, more than being just a way of registering and demonstrating developments in computer graphics modeling of topological surfaces, successive animations of the sphere eversion became a way of conceiving of and constructing a new kind of topological observer, one attuned to a qualitative rather than quantitative investigation of space.

"The Theorem Was, 'Wow! We Can Understand It'"

In 1964, Anthony Phillips was completing his dissertation at Princeton University under the supervision of mathematician John Milnor when he devised a new proof that spheres could be turned inside out.[40] Like Smale, Phillips was able to show that a regular homotopy exists but not what it might look like. This was not great news for Phillips, as he was already familiar with Smale's result. However, once he proved it himself he became fascinated with the question of how to visualize the process. Phillips recalled that while no solution was known at the time, he had heard "a legend of sphere eversion." According to the rumor, soon after Smale's proof was published, topologist Arnold Shapiro had found an explicit solution and, before he died, had divulged it to several mathematicians, among whom was René Thom. Phillips decided to write Thom a letter. In June 1964, he received a reply confirming that the legend was true. The letter was short, two pages of text with some elementary line diagrams describing Shapiro's solution.

The central idea behind Thom's sketches was the realization that a well-known familiar surface, known as Boy's surface, can serve as a point of departure. Named after Werner Boy, Boy's surface is an immersion of the real projective plane in regular three-dimensional Euclidean space.[41] This fact was meaningful because what mathematicians call the *double cover* of the real projective plane, and hence Boy's surface, is a sphere.[42] Leaving the exact mathematical meaning of these statements aside, what is important to note is that this realization effec-



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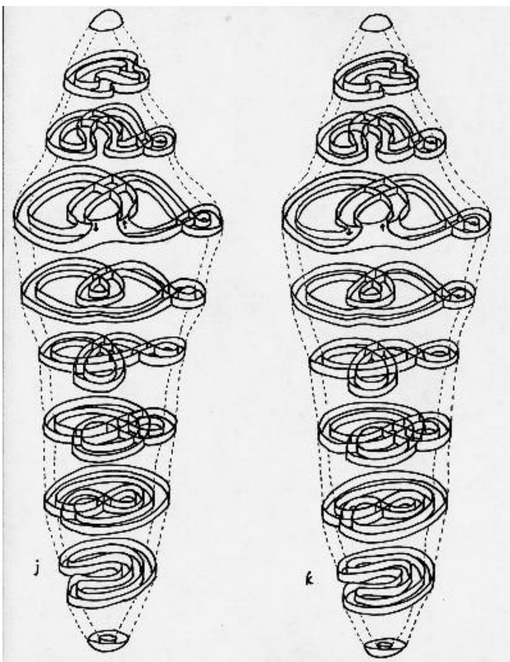
qui sont tous deux régulièrement isotopes au cercle standard dans le plan . On voit donc que la fonction (z) présente sur la sphère (W) six points critiques : deux minimum , deux points-col , deux maximum . Il suffit de montrer qu'on peut , par homotopie régulière , éliminer un couple (minimum point-col), conformément au modèle local ci-contre (résorption d'une "bosse") ; si l'on peut de même éliminer l'autre point-col avec un maximum , par exemple , dans une isotopie ne faisant intervenir que le demi-espace z > c , on aura finalement déformé (W) en une immersion qui ne comporte plus pour (z) que deux points critiques : un maximum et un minimum ; dans le diagramme donné ci-dessus , on voit que l'immersion résiduelle n'est autre que le plongement standard (Y) . Ceci achève la construction .

Il serait fort intéressant de savoir s'il n'existe pas de solution plus simple pour cette déformation ; on aimerait également connaître la complication minimum présentée par la courbe de self-intersection au cours de la déformation (Points triples etc).

Je serais heureux de recevoir vos commentaires sur la question , et vous prie de recevoir , cher Monsieur Phillips , l'expression de mes meilleurs sentiments .

René Thom
Institut des Hautes Etudes Scientifiques
Bures s/Yvette
Seine et Oise
France

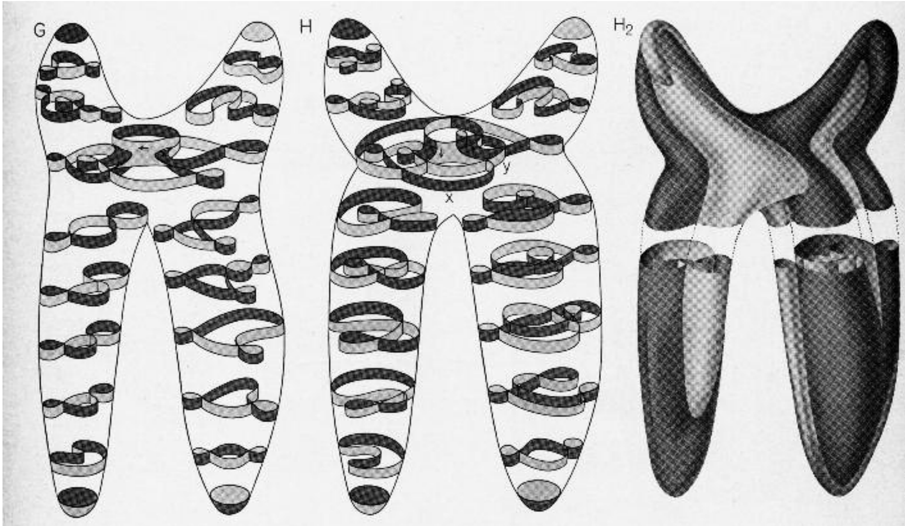


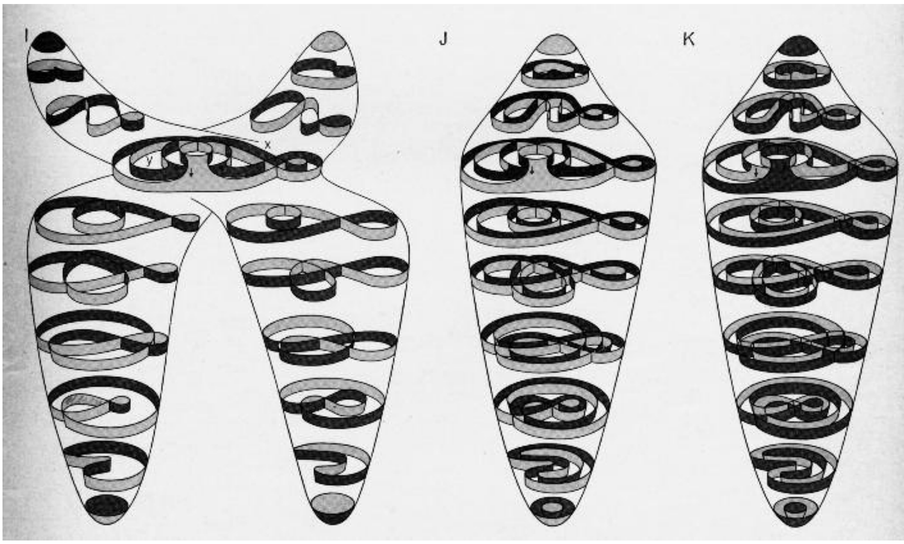
tively translated the problem of finding an explicit eversion to finding a way to deform the double cover of Boy's surface (whose geometry was relatively well known) into a sphere.[43] This was still a difficult task, yet a tractable one. In his letter, Thom described how such a deformation should proceed, focusing on key moves in the transformation. While he included two sets of line diagrams, Thom's explanation was mostly verbal and relied on familiarity with the geometry of Boy's surface along with various tools of differential topology.

Phillips was able to follow the line of reasoning laid out in Thom's letter, but he wanted more. The letter gave Phillips an idea as to a general procedure; it explained how to avoid the creation of creases and pinch points, but it did not provide him with a complete description of the process from start to finish. Phillips set to work. During the summer he had a job at a government-sponsored research lab in Cambridge, Massachusetts, but in his free time he relentlessly worked on the problem. He began drawing illustrations of the process. One day, as he was going over his drawings, he "found a sketch, a way to understand it."[44] Thus, while laboring on his drawings as he tried to translate Thom's letter into a coherent representational scheme, Phillips first felt he grasped the problem.

Phillips not only had to comprehend the various steps in the deformation, how the surface twisted and turned and what curves were created in the process; he also had to devise a way to represent the process. This was no trivial matter. Phillips "sliced" the sphere into a series of parallel rubber bands, or cross-sections, that enabled him to follow simultaneously both the external and internal structure of the sphere throughout the deformation. The illustrations he produced served as a visual guide to the transformation.

Realizing that the problem might have an appeal beyond the mathematical community, he contacted Martin Gardner, who edited a popular column on math-





ematics for *Scientific American*. Gardner expressed interest, and "Turning a Surface Inside Out" was published in May 1966. The article provides a short general introduction to the field of differential topology, but its heart is the numerous illustrations that are included. By following the illustrations from A to S, the reader is instructed in how the deformation unfolds. As Phillips explains in the text, "it is possible to understand the entire deformation by interpolating the missing parts of the surface at each stage and checking that the changes in the ribbons depicting various sections fit together coherently."[45] The difficulty of the task the reader is asked to perform is hard to overemphasize. Although the illustrations are elaborate and highly detailed, much is left to the reader's imagination. The illustrations function more as visual clues that direct the reader in how to construct his or her own image of the eversion. They do not show, but rather tell.

Phillips chose to publish in *Scientific American* "because it wasn't a mathematical result. It was just a visualization . . . there was no theorem." As he said, "the theorem was, 'Wow! We can understand it.'"[46] That theorems are the most important currency among mathematicians is a truism. Phillips's demonstration in *Scientific American* was accepted as a demonstration only because mathematicians

Anthony Phillips. "Turning a Surface Inside Out," *Scientific American*, 1 May 1966. Steps 7–12.

(following Smale's proof) already believed the process was possible. Certitude was the outcome not of visual demonstration but of formalistic deduction.

Yet, Phillips's remark points to an important distinction he holds between knowing something to be true and understanding why it is true. Phillips explained the motivation behind his work: "We knew it can be done . . . as a particular consequence of Smale's work, this double cover could be changed by regular homotopy into a regular round sphere." He then added, "Okay, but the question was how to explain it, how to visualize it, and understand it."[47] All the mathematicians who worked on the problem in the following decades repeated this entanglement of understanding with visualization.

How should we think about the images Phillips produced? What sort of objects did they illustrate? And, how did they function in relation to knowledge? Despite his conviction to the contrary, the eversion Phillips depicted in his illustrations differed from the one Shapiro discovered and described to his colleagues. The insights Thom described in his letter to Phillips diverged from Shapiro's. Thom was repeating an idea that had already appeared as an aside to a 1961 article by mathematician Nicolaas Kuiper, but the solution had not previously been worked out from beginning to end. To conceive of Phillips's illustrations simply as representations of an already fully formed theoretical idea, or as instruments that aid in formalizing understanding, would therefore be incorrect. Instead, the drawings and the solution were mutually constitutive. The concrete and the abstract were not posed one against the other but worked in tandem.[48]

"A Mental Movie"

The publication of Phillips's article drew more mathematicians to the problem, one of whom was Nelson Max. As a graduate student in the department of mathematics at Harvard University, Max came across Phillips's original drawings even before they appeared in print. He took the drawings home with him and worked into the night, struggling to follow the various steps of the construction. Max later explained his difficulty:

> You could sort of see the different surfaces sliced open that way, but you had to fit them together continuously in your brain to figure out how the whole surface would have looked if you hadn't cut the slices that way, and then you had to interpolate them in time to see how the deformation would move.[49]

His difficulty—an inability to *interpolate the images in time*—prompted Max to try to produce an animation of the problem.

During the summer of 1967, Max attended a conference on mathematics and physics at the newly established Battelle Seattle Research Center.[50] Among the participants were mathematicians Smale, Thom, and Bernard Morin, and physicists Bryce DeWitt and Marcel Froissert. During the conference, Max was introduced to the work of Morin, who, following the work of DeWitt, was devising a new procedure to turn the sphere inside out.[51] Unlike Phillips's demonstration, Morin's did not involve a series of illustrations. Rather, in order to communicate his ideas to his colleagues, Morin constructed a series of clay models of key stages in the deformation. Morin has been blind since the age of six. In an interview several years ago, he explained, "our spatial imagination is framed by manipulating objects . . . you act on objects with your hands, not with your eyes. So being outside or inside is something that is really connected with your actions on objects."[52] A tactile rather than visual sensation was instrumental to Morin's solution.

As James Griesemer notes, "The tactile, muscular, kinaesthetic experience of working with a [three-dimensional] model brings experience to the mind in more and different ways than the merely visual."[53] Mathematician Silvio Levy similarly argues, "the fact that he [Morin] was one of the first people to understand how a sphere can turn inside out is both a tribute to his ability and a convincing proof that 'visualization' goes far beyond the physical sense of sight."[54] Morin's solution was manifested simultaneously both mentally and materially. This was not an abstracted topology but a tangible one. Separating the formal description of the solution from the three-dimensional clay models is an act done only in hindsight. As with Phillips's drawing, the solution fully depended on the medium. The clay models were not a post hoc representation of an already well-conceived solution but were objects *literally* to think with.[55]

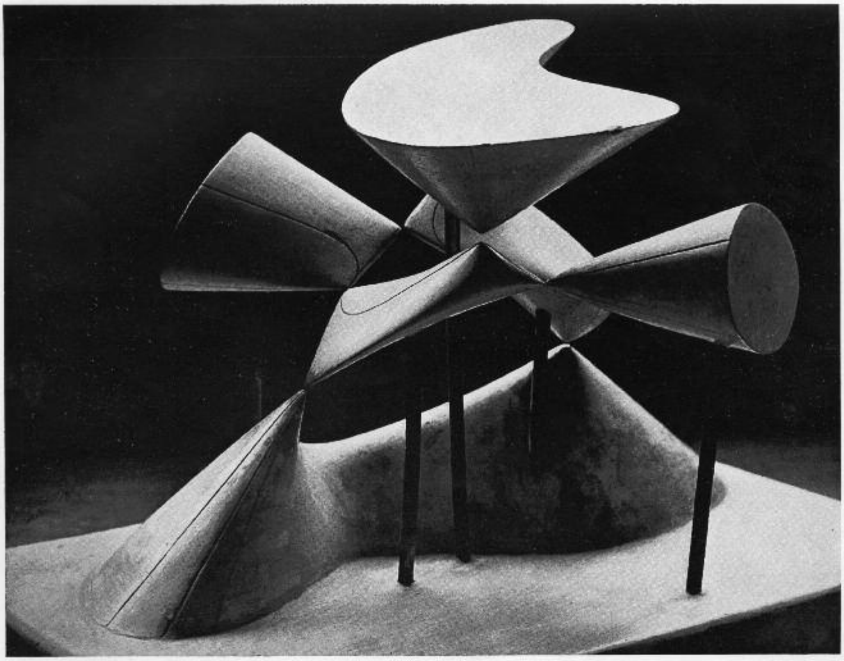
Following their meeting, Max decided to depict Morin's rather than Phillips's version of the eversion.[56] He first tried his hand in Claymation, but in 1968 he came across his first computer graphics station and decided to produce a computer animation of the problem.[57] The newly available technology, Max reasoned, would enable mathematicians not only to attain a sequential apprehension of the eversion but a *temporal* perception of it. Max applied to the National Science Foundation to fund his work, and in 1970, with the help of the Education Development Center in Newton, Massachusetts, the Topology Films Project was born.[58] When he began, Max had no training in computer graphics, nor did he have any training in film production. He faced two technical problems: first, how to define the homotopy to the computer; second, how to determine an efficient computation method for each frame through appropriate hardware. Six years would pass before Max completed the film.

In the summer of 1971, Max traveled to the University of Cambridge to use the MultiObject system, a new interactive graphics program developed by mathematician Andrew Armit. The system enabled the user to input a description of the object he or she was interested in displaying as a combination of more basic shapes (e.g., cylinders, circles, and squares). Max spent more than a week trying to input into the computer a description of a single stage of the deformation but soon learned that the task was much more difficult than he had imagined. Talking to other users of the system, he found that the system worked successfully only when the input was based on measurements of actual three-dimensional objects. That is, no one had yet managed to use the system to render an object on the computer screen following only a formal description of the object, whether mathematical or textual. Luckily for Max, when he returned to the United States he found that Pugh had the solution to his problem.

Pugh was determined to understand the geometry behind Smale's proof. He read Phillips's *Scientific American* article but was left unsatisfied. However, unlike Max, instead of resolving to animate the problem, Pugh became (in his words) "obsessed" with building a model of a stage in the deformation. If for Max the limiting factor of Phillips's demonstration was its inherently static and discrete nature, for Pugh the problem was that the demonstration was two-dimensional. A flat representation, regardless of how accurately and masterfully it was rendered, would always, as far as Pugh was concerned, be inadequate because it failed to capture the three-dimensional character of the problem. "You couldn't turn the picture around," Pugh explained. "A model, you can pick it up, you can look at the bottom, the top, and you can see how it is organized." Like Phillips before him, Pugh became "convinced" that in order fully to conceptualize the problem, he first had to make it.[59]

During a sabbatical at the Institut des hautes études scientifiques in 1969, Pugh struggled to construct a model of the halfway stage of Morin's eversion.[60] He first tried to build it out of plaster, then moved to papier-mâché, then during a summer in Nice tried his hand at fiberglass. Lacking prior training in the fine arts, Pugh was repeatedly frustrated in his efforts. The plaster model quickly became bulky as he added more and more layers, the papier-mâché was unstable, and working with fiberglass turned out not only to be harmful to his health but much harder to mold than anticipated. At the end of the year, Pugh finally came up with a working solution: he realized the medium he sought was the very wire he had been using as a frame in his other attempted models.

When Pugh returned to Berkeley, the department of mathematics relocated to



SURFACE DE KUMMER A SEIZE POINTS DOUBLES, DONT HUIT RÉELS (PHOTO MAN RAY)

a new building, Evans Hall. To celebrate the occasion, the department held a contest to decorate the entrance hall. Pugh won and set about constructing additional models (the first model was of a perfectly round sphere and the tenth was Morin's halfway model).[61] The decision to display Pugh's models fits within a long mathematical tradition of exploiting the tension between the aesthetic and the pedagogical values of mathematical models. At the turn of the century, many mathematical centers across Europe and North America owned collections of three-dimensional plaster and string models.[62] The University of Göttingen, a major site for the production of these models, even had a separate room adjoining the mathematical library dedicated to their exhibition; they were also frequently deployed in the classroom. During the first few decades of the twentieth century, and in conjunction with the rise of mathematical modernism, these models fell out of fashion and were relegated to storage rooms and dusty cabinets.[63]

During this period, constructivist artists, including Naum Gabo, and surrealist artists such as Max Ernst and Man Ray maintained an interest in mathematical models. Ray, who visited Poincaré's collection in Paris, completed a series of photographs of these models that he titled *Mathematical Objects*.[64] He noted that the "shapes were so unusual, as revolutionary as anything that is being done today in painting or in sculpture."[65] The sculptural nature of Pugh's models elicited similar astonishment among his peers. However, unlike fin de siècle sculptural models that represented fixed and static geometrical surfaces, the arrangement of Pugh's models in sequential order emphasized their topological (rather than their geometric) origins. They indexed a process, not an object. As such, whereas Ray commemorated found mathematical sculptures in photographs, Max resolved to capture Pugh's models on film.

Pugh recalled that as he was completing his work on all ten models, for exactly one day he was finally able to close his eyes and "see" the deformation of the sphere from beginning to end. He placed the ten models in a row on his work-

Opposite: Man Ray. "Surface de Kummer a seize points doubles, dont huit réels," *Cahiers d'art*, 1936.

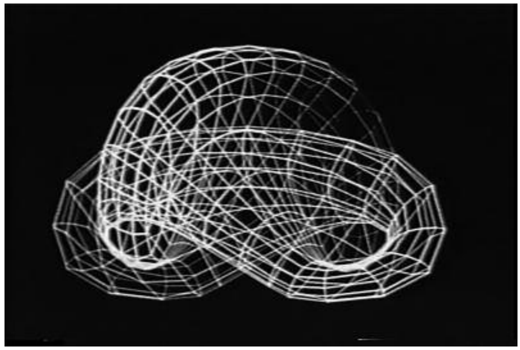
Left: *La faim/Hunger*, dir. Peter Foldès, 1974.

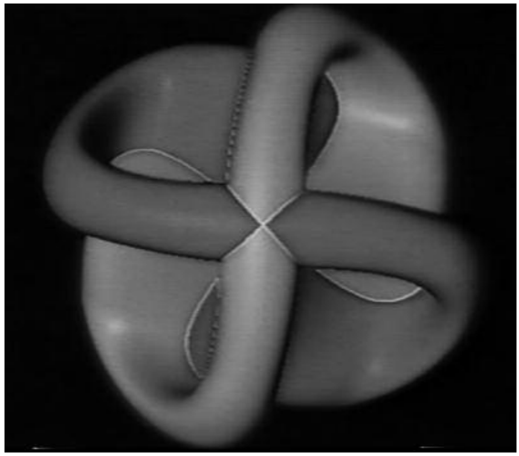
bench and moved among them. "They were big enough that you could run your hands along them . . . you could feel the smoothness to it, kind of stroking them, and that kind of physical interaction with the object is something that when you close your eyes you can still appreciate."[66] The touch of the hand, the smoothness of the surface, and the motion of the body as it proceeded from one model to the next reproduced, for Pugh, the movement of the deformation in his mind's eye. At that moment, tactile and kinesthetic experience both grounded and forwarded theoretical understanding.[67]

When Max found out about Pugh's models, he immediately recognized that they provided a solution to his first technical problem. Over the course of a few weeks, Max rented a car for the first time in his life and drove the models, two at a time, across the Bay Bridge from Berkeley to Stanford. Using masking tape, he divided the surface of the models into quadrilaterals and triangles, measuring their location. He cut arrows out of cardboard and glued two at each vertex to represent the two tangent planes at that point on the surface. In order to measure the coordinates of the tip of the cardboard arrow, Max stuck chewing gum to the end of a string and held it to the tip of each arrow, noting on graph paper the tip's height and location. He entered the information interactively into a PDP-10 computer using a program he wrote that was inspired by Armit's MultiObject system. When he finished, Max had finally succeeded in modeling on the computer the ten stages of the eversion. Finally, in order to animate the eversion, Max programmed the computer to interpolate between the static stages.

Max was not the first to interpolate between images to produce computer-generated animations. One of the first computer animation films, *La faim/Hunger*, was produced by the National Film Board of Canada using graphic interpolation (it won the Special Jury Prize at the 1974 Cannes Film Festival). The fundamental difference between *La faim* and Max's technique was that the key frames in *La faim* consisted of artist's drawings, whereas in Max's film the stills were defined by the physical measurement of Pugh's models.[68] The origin of the film in the actual measurement of physical objects is evident in the final animation, which calls upon a haptic rather than an optic sensitivity in its viewers.

Max resolved to depict the deformation both as a wire model and as a colored shaded surface. Using a graphic display processor, which was developed at Carnegie Mellon University and worked in conjunction with a PDP-11, Max was able to create a sequence of wire frames, which he used for the first half of his movie. Creating a shaded surface proved to be complicated.[69] The surface of the





sphere was input to the computer as a collection of patches (eighteen rectangular and eight triangular), which were further divided into six-by-six arrays to create smooth surfaces. At Case Western Reserve University, Max used Evan and Sutherland's LDS2 animation machine in conjunction with a PDP-11 to create a continuous family of shaded opaque surfaces.[70] Each frame of the film had to be computed independently with the computer controlling the advancement of the film in the camera.

Throughout, Max had to solve technical problems, such as how to ensure that the surface appears smooth at the edge where two patches meet, as well as how to determine the correct coloring of a patch that was in the process of turning away from the viewer's point of view. By the time he presented his work at the SIGGRAPH conference in 1976, his transition to computer graphics was complete; he had become an expert in scientific visualization. As such, his career trajectory exemplifies much research on the sphere eversion problem, which became over the years "the 'teapot' of visualizable geometry." In trying to utilize computer graphics to study such topological problems, mathematicians relied on existing tools but also constructed (most often in collaboration with computer scientists) new programs uniquely suited to geometric modeling.[71]

Turning the Sphere Inside Out was novel because it represented the first complete, temporally continuous visual demonstration of an eversion. The idea of smooth movement may already be built into the definition of the problem, but until Max's film was released the movement involved in the homotopy was implied rather than realized. Phillips, Pugh, and Max all describe their individual desire to be able to visualize a deformation of the sphere from beginning to end. Only with the release of Max's film were they able to see it. Computer animation thus enabled mathematicians to approach mathematical objects continuously rather than discretely.[72] Consequently, as visualization techniques changed, so did what constituted geometrical apprehension. Whereas the impulse for mathematicians such as Phillips and Pugh was imaginatively to

Opposite, top to bottom:
***Turning a Sphere Inside Out*, dir. Nelson Max, 1977. A vector graphics display.**

***Turning a Sphere Inside Out*, dir. Nelson Max, 1977. A fully shaded opaque rendering.**

***Turning a Sphere Inside Out*, dir. Nelson Max, 1977. A shaded polygonal rendering.**

understand the geometry that was implied by the theorem, by the time Max completed his film, the homotopy itself began to be conceived of as a movie.

For example, in 1994 a team of mathematicians and computer graphics specialists produced *Outside In*, a computer animation of a sphere eversion based on the ideas of Fields medalist William Thurston.[73] Thurston explained his motivation for developing yet another solution to the problem: "Pugh's models were beautiful, well-made, and each transition was clear. . . . It was also hard for me, even after carefully studying them and following the sequence step by step, to assemble them into a coherent story or a *mental movie*."[74] In narrating his own study of the eversion, Thurston did not refer directly to Max's film. However, his desire for a "mental movie," which he equated with having an "alive and direct" understanding of the problem, points to a transformation in the imaginative practices mathematicians brought to bear on the problem. As Thurston noted, *Outside In* represented an effort to use a "different, emerging medium, to communicate a dimension of insight different from that typically conveyed by a mathematical paper."[75] Mathematical animation had transformed from a means to an end; it became a way of not only representing but of conceptualizing the problem.

"Something You Can Recreate in Your Mind"

Max accomplished what he had set out to do—namely, to produce a computer animation of a sphere turning inside out from start to finish. However, he summarized the enterprise as "unsuccessful." He explained, "I want[ed] people to believe that you can turn the sphere inside out because it happens in front of their eyes. I think that I convinced people of that." And yet, he added, even after "seeing the film more than once . . . it is really hard to go away . . . with something you can recreate in your mind . . . [something] that you really understood because you could reproduce it."[76] Max worked on the film for six years, devoting hours of labor and intellectual energy to its production. When the film came out, it stood at the forefront of what was then possible to illustrate using computer animation. Why then did he judge the project unsuccessful? The answer is discernable in the difference Max draws between the film's goals: on the one hand, to convince or demonstrate and, on the other hand, to explain. Understanding, not belief, is what Max hoped to foster in his audience.

According to Max, for the film to be successful it had to elicit in its audiences a recapitulation of what it performed. Audiences must be able, as Max said, imaginatively to "recreate" or "reproduce" the topological transformation themselves. At least in part, the perceived failure of the film was due to the difference between the inherently perspectival nature of the computer display and the topo-

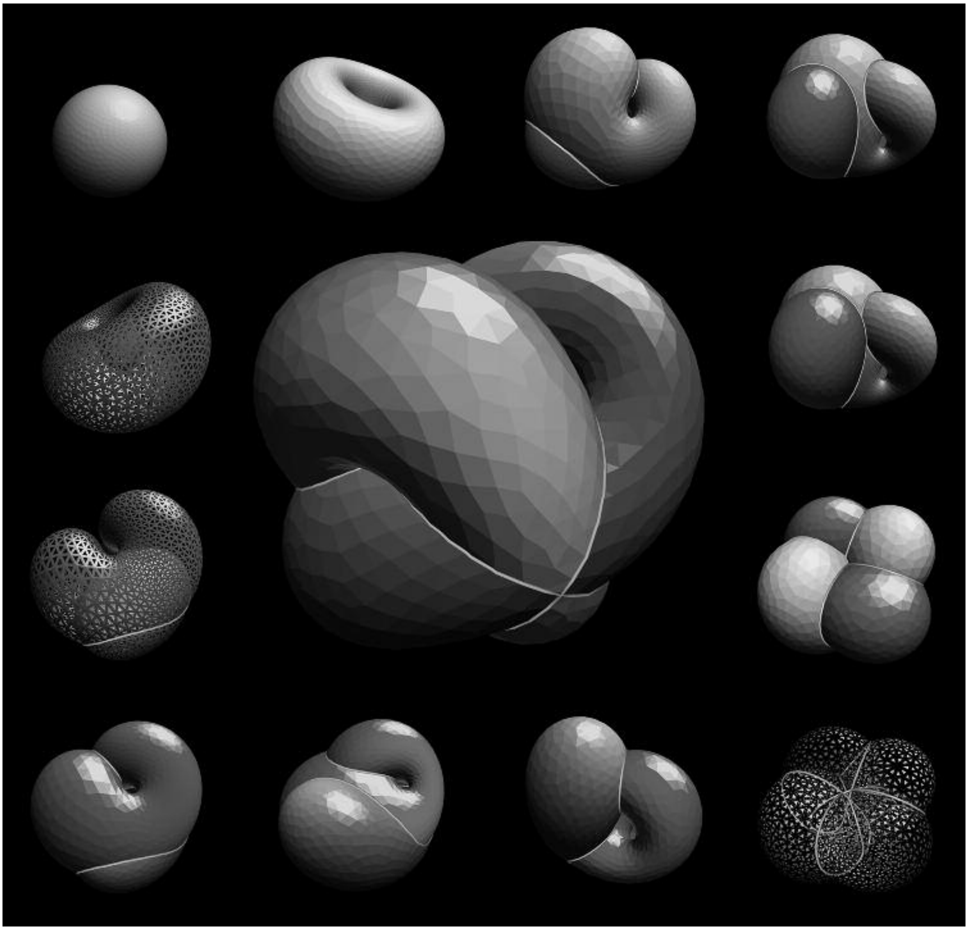
The Optiverse, dir. John M. Sullivan, George Francis, and Stuart Levy, 1998.

logical space it sought to emulate. The animation fostered haptic vision, but it relied on a geometric perspective that privileged a fixed point of view from which the observer could watch the demonstration as it unfolded in the space in front of him or her. Moreover, sight by itself was insufficient exactly because it erased the body's mobility. Understanding, however, required a more active engagement of the viewer. Aided by novel interactive computer technology, increased computational power, and advancements in virtual reality, the mathematicians who came next set out to foster just that.

The Optiverse came out in 1998. The work of a team at the University of Illinois at Urbana-Champaign comprising mathematicians George Francis and John Sullivan and computer graphics specialist Stuart Levy, it demonstrates yet another solution to the eversion problem. What distinguishes the eversion depicted in this film from its predecessors is that it was computed automatically through a process of optimization. The *minimax eversion*, as it was named by its creators, did not rely on a geometrical insight by some mathematician but instead on increases in computational capabilities. Commenting on the difference between their method and the ones that came before it, the creators of *The Optiverse* write, "for all of these eversions, their designers had to know *a priori* what they should look like, and then search for formulas capturing this behavior."[77] Theirs, they explain, was instead achieved by building exploratory tools.

The creators of the minimax eversion did not try to program their geometrical intuitions onto computer screens. Rather, using the Surface Evolver, a program written by mathematician Ken Brakke for studying surfaces under various constraints, they instructed the computer to minimize what is known as the elastic bending energy of the surface and then sought to depict on screen what an eversion constructed by such principles would look like. In *Turning the Sphere Inside Out*, Max used available computational capabilities to interpolate between frames, but the eversion itself was well conceived before its realization. In contrast, the deformation depicted two decades later in *The Optiverse* was determined *computationally*.[78] Before they saw the homotopy unfold on the computer screen before their eyes, the creators of *The Optiverse* had no idea how it might transpire. The medium and the solution were one and the same.

In their analysis of the use of computer animation in the life sciences, Christopher Kelty and Hannah Landecker argue that biological animations should be understood not "in relation to the real" but "*in relation to knowledge*."[79] The same undoubtedly applies to *The Optiverse*. The film is not indexical of some external phenomenon but functions in relation to mathematical theory. Even



more surprising, the film constitutes what counts as knowledge in the act of its realization. Like the solutions that came before it, the demonstration of the deformation realized in *The Optiverse* did not precede its presentation but instead completely depended on its digital realization. The creators of the film were able to program their theoretical presuppositions, but the work of articulating a solution was fully delegated to the computer. Thus, more than an animation of a theory, *The Optiverse* can be described as a *theory, animated*.

Like Max before them, the creators of *The Optiverse* had to contend not only with how to program the eversion but also with how to display it using available graphic technologies. Instead of building upon already existing software, they wrote their own real-time interactive computer animation software, AVN. That software, which first ran on SGI workstations and later on laptops, controlled the visualization and animation of the eversion that was computed with the Surface Evolver. In an article detailing some of the main features of their software, the creators note that "the mathematician shares one significant attribute with the artist: both prefer to make their own tools rather than limit their creativity to the constraints of ready-made tools available 'off the shelf.'"[80] What creative freedoms were the creators of *The Optiverse* hoping to achieve? Two main features distinguish their work from that of their predecessors: first, the possibility of toggling among several rendering techniques; second, a break from a fixed point of view. Both of these capabilities, which they implemented in their program, assumed an active user. More than a filmic demonstration, they sought to provide users with an exploratory space in the hope of teaching them how to inhabit topological spaces *imaginatively*.

The main difficulty in comprehending the homotopy, according to the creators, was the phenomenon of intersecting surfaces moving in time. In order to overcome this difficulty, they represented the eversion using various rendering techniques. Employing color shading, transparency, and fragmentation, the surface of the sphere is altered throughout the deformation, revealing and obscuring in turn various features of the eversion. The creators were unable to escape the perspectival nature of the display screen, but they tried to use various visualization techniques to furnish the viewer with topologically meaningful information. This does not mean that aesthetic criteria did not guide their choices. In developing their software, they borrowed the same "artistic conventions" that anatomists, who confront similar difficulties, have used for centuries, appropriating them to fit their topological needs. In the process, the "cut away," "the window," and the "excised organ" become "conformal warping and shaping," "geometrically meaningful rendering and coloring," and "isolating details."[81]

They also depart from prior films in deciding to break away from a fixed point of view. The "script" was not set in advance. Rather, the user is able to manipulate the deformation in real time. In order to understand the transformation, the user, in their view, needs to project his or her agency onto the process. They give users three modalities with which to watch the deformation: as a manipulable object placed in a space in front of them, as a static object they can "move" around, and finally as an object they can inhabit from within. In the first case the point of view remains constant, but the sphere can be turned a full 360 degrees at any given moment throughout the transformation. In the second, the "camera" can move around the sphere, observing it from any angle. Interacting with the transformation, choosing its speed and perspectival unfolding, users can both literally and figuratively grasp the transformation.

The third case, however, presents the most challenging point of view to the viewer and most actively challenges the perspectival nature of the display. At any point during the transformation, users can choose to "step inside" the sphere and observe the transformation as it would be perceived by a subject who was "inside" the sphere. In an article describing their work on the film, Francis and Sullivan draw an analogy between a two-dimensional window in a wall, which "lets us look *into* a world, while blocking out the distracting periphery," and an equivalent "window" in three dimensions. The program, they explain, creates a "*positive* 3D clipping box" that "blocks out the surrounding material, enabling us to look at convoluted internal structure and processes, one part at a time, and from all sides."[82] The deformation still takes place in virtual space, but its producers now aim for the body of the mathematician to be projected into the deformation rather than to observe it from the outside.

When run in a Cave Automatic Virtual Environment (CAVE), this "positive 3D clipping box" is especially powerful, as it places the body of the viewer "inside" the sphere. Unlike the user watching a computer screen, in the CAVE the sphere appears to evert around the user. Lifting one's head up, moving it to the right or the left, the images the user sees readjust to reflect his or her changing point of view. Stepping forward, the user is given the illusion of passing through the surface of the sphere as what is "hidden" behind him or her comes into view. Kinesthetic and visual perceptions are joined. Banished by the computer, the body now returns, moving through the CAVE and the sphere, manipulating its surface.[83]

This is not to suggest that this process is straightforward. Francis and Sullivan acknowledge that only through training can the user learn how to navigate such a space. "Guiding this 3D probe about—much like a pre-literate child will follow words with her finger—we can explore the sphere eversion."[84] That is, only through a process of inculcation can users learn how to navigate "inside" a sphere. To what degree the program succeeds in providing users with an appreciation of topological space is impossible to determine. Yet the software clues us in to how the programmers conceived of topological observation. In order *imaginatively* to inhabit a topological space, they ask the user to approach it not from the outside but from the inside.

While they celebrate the opportunities afforded by new technological innovation, Francis and Sullivan recognize that it comes with its own limitations. In 2003, the two mathematicians published an article expounding on the technical aspects of their work and the representational possibilities occasioned by advancements in computer graphics. Nonetheless, the authors conclude their article by describing not virtual but "sculptural" models. The former, they note, have many advantages over the latter, but "despite many advances in virtual reality, immersive environments, and holography, real physical models still have other advantages." That is, "they give a sense of concreteness which is hard to find in the virtual world."[85] The virtual and the material do not need to be positioned against one another. Rather, they are here complementary strategies whereby one gives what the other takes and vice versa. In a complete reversal of events, the computational data from the program was used to print three-dimensional stereolithography models of key stages in the eversion. Francis and Sullivan presented these models to Morin.

Conclusion

During an interview, Phillips summarized the history of the sphere eversion problem, noting that "the real research is into human perception." He explained, "the whole point is to have people with our evolutionarily constructed three-dimensional minds understand this process, which is very hard."[86] Since the nature of the eversion challenged the limits of sensory apprehension, his own topological investigations in the 1960s, he suggested, were also explorations of human perception. Writing a few years before Phillips began drafting his illustrations, Maurice Merleau-Ponty drew a similar connection between topology and perception. If "Euclidean space is the model for perspectival being," he wrote in his posthumously published notes, then "topological space, on the contrary, [is] a milieu in which are circumscribed relations of proximity, of envelopment, etc."[87] Topology, Merleau-Ponty proposed, offers a model for human perception that breaks away from Cartesian epistemology. Instead of positing a knowing subject and an objective world exterior to that subject, topology reaffirms the whole body as a site of perception.

Merleau-Ponty's appeal to topology might be read as a useful metaphor with which to rethink theories of embodied perception. Yet for topologists themselves, the sphere eversion became a topological laboratory with which to rethink the limits of perception. In trying to attain a more comprehensive grasp on the problem, one that went beyond theoretical understanding, topologists were forced time and again to attune their senses to the topological phenomenon of the sphere eversion. Building upon a host of technological mediations, they aimed to bring all their perceptual faculties to bear on the problem, from optic, to tactile, to haptic, to kinesthetic sensation. Thus, theories of perception were not merely side effects but rather were at the heart of topologists' attempts to comprehend the sphere eversion. Explorations of topological spaces were by their nature also inquiries into human perception, its extensions, mediations, and limits.

Merleau-Ponty suggests that a faithful mode of embodied perception requires a topological conception of space: "I describe perception as a diacritical, relative, oppositional system—the primordial space as topological (that is, cut out in a

Bernard Morin holding a model created by Stewart Dickson with a 3-D printer, 2000. Courtesy John M. Sullivan.

total voluminosity which surrounds me, in which I am, which is behind me as well as before me . . .).”[88] Instead of a metric Euclidean conception of space, topology calls for a qualitative appreciation of space. Topological space is, as such, a site of intentional embodied experience prior to any reflective thought. Space does not extend in front of but expands around the subject, seeing and being seen, touching and being touched.

Perhaps, then, theories of perception and topological theories should be examined in tandem. Both perception and topology function in relation to perspectival Euclidean space. As a modern mathematical field, topology broke away from projective space by considering exactly those qualities of figures that remain fixed under smooth transformations, regardless of distance. A phenomenological theory of perception similarly sought to move beyond perspectival space and its assertion of vision as the primary mode of perception. That is, both are reactions to the union of vision and Euclidean geometry.[89] Yet, in grappling with topological spaces beyond abstract formalism, mathematicians returned to media governed by projective space, whether print, three-dimensional models, or computer graphics. In the process, topological investigations became a way for mathematicians to query bodily perception. So too for phenomenologists following Merleau-Ponty: reflecting upon the nature of human perception demands a spatial modality that extends beyond a strictly quantitative notion of space. Instead of considering the two as separate domains that are brought to bear on one another in a post hoc fashion either metaphorically or operationally, they can be understood better as two parallel developments that inform one another.

For this reason I suggest *manifestation* as the joint between theories of perception and theories of topology. Offering ways of engaging with mathematical entities that stretch beyond the symbolic, mathematical manifestations do not function within a referential framework.[90] Beyond representation, manifestation emphasizes not a simple one-way relation between a fully formed idea and its material instantiation but the fact that the two are mutually constitutive. Over the years, mathematicians working on the problem sought to make evident to the eye and the hand what were otherwise formalized and well-defined mathematical structures and procedures. That is, they operated in relation to (topological) knowledge, not in relation to the “real.” In the process, the thoroughly mediated approach to mathematical sense-making that emerged from applying computer graphics to topology demanded a new mathematical observer, one attuned to perceiving topological rather than geometrical spaces.

More than referring to the specific topological problem at hand, the process of turning inside out or outside in also offers a model of mathematical perception that places mathematicians' bodies at its center. Instead of adhering to a stance that posits the abstract against the concrete and the symbolic against the material, *inside out* points to the traffic between the two and invites a phenomenological inquiry into mathematical practice. Trained mathematicians' bodies are the junctions, sutures, and conduits that transduce mathematical entities, manifesting them both mindfully and materially.[91] These processes are not binary. Instead, mathematical manifestation is one way that mathematicians evert the two. The mathematicians who have worked on the sphere eversion have actively sought to breach gaps between the sensed and the sensible.

Notes

It is a great pleasure to thank the many mathematicians whose work inspired this paper. I thank George Francis, Nelson Max, Anthony Phillips, and Charles Pugh for generously sharing their time and stories with me. Lorrain Daston, Stefan Helmreich, David Kaiser, Florian Klinger, and Sophia Roosth all read drafts of this article and provided invaluable feedback. Finally, I wish to extend my gratitude to the editors of *Grey Room* for ushering this article into publication.

1. Maurice Merleau-Ponty, "Eye and Mind," trans. Carleton Dallery, in *The Primacy of Perception and Other Essays on Phenomenological Psychology, the Philosophy of Art, History and Politics* (Evanston, IL: Northwestern University Press, 1964), 164.

2. Nelson L. Max, dir., *Turning a Sphere Inside Out* (1976; Wellesley, MA: A K Peters, 2004), VHS.

3. *Eversion*, which according to the *Oxford English Dictionary* means "the action of everting or turning (an organ or structure) inside out," is distinguished in mathematics from *inversion*, which denotes a way of transforming a shape according to a set of specific transformations.

4. Max, *Turning a Sphere Inside Out*.

5. Max, *Turning a Sphere Inside Out*. Years later, mathematician Silvio Levy explained that to be able to visualize the geometric construction following Smale's proof was "akin to describing what happens to the ingredients of a soufflé in minute detail, down to the molecular chemistry, and expecting someone who has never seen a soufflé to follow the 'recipe' in preparing the dish." Silvio Levy, ed., *Making Waves: A Guide to the Ideas behind Outside In* (Wellesley, MA: A K Peters, 1995), 31.

6. Throughout this article I use *to see* to denote the particular desire for a fully sensorial and cognitive comprehension expressed by the mathematicians about whom I write. The conflation of seeing and knowing by itself is not surprising, as it dates back to the ancients, to authors such as Plato. For the mathematicians I study, seeing is not limited to the sense of sight. Furthermore, it entails a formal mathematical way of knowing. See Krzysztof Pomian, "Vision and Cognition," in *Picturing Science, Producing Art*, ed. Caroline A. Jones and Peter Galison (New York: Routledge, 1988), 211–31.

7. For centuries, astronomy, harmonics, optics, and statistics were included under the rubric of mathematics. As such, an exhaustive description of the role of material artifacts in mathematics is impossible in these pages. Focusing on geometry, which is the only classical field that (from the modern perspective) still belongs to mathematics, the role of images, models, and instruments up until the twentieth century is widely treated in the secondary literature. See, for example, Jim Bennett, "Geometry in Context in the Sixteenth Century: The View from the Museum," *Early Science and Medicine* 7, no. 3 (1 January 2002): 214–30; Fokko Jan Dijkstehuis, "Moving around the Ellipse: Conic Sections in Leiden 1620–1660," in *Silent Messengers: The Circulation of Material Objects of Knowledge in the Early Modern Low Countries*, ed. Sven Dupré and Christoph Lüthy (New Brunswick, NJ: Transaction Publishers, 2011), 89–125; Stephen Johnston, "John Dee on Geometry: Texts, Teaching and the Euclidean Tradition," in "John Dee and the Sciences: Early Modern Networks of Knowledge," special issue, *Studies in History and Philosophy of Science Part A*, 43, no. 3 (September 2012): 470–79; Megan McNamee, "Picturing Number in the Central Middle Ages" (Ph.D. diss., University of Michigan, 2015); Herbert Mehrtens, "Mathematical Models," in

Models: The Third Dimension of Science, ed. Nick Hopwood and Soraya de Chadarevian (Stanford: Stanford University Press, 2004), 276–305; and Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Cambridge, UK: Cambridge University Press, 2003).

8. Here I do not include simple line diagrams, whose use in mathematical research continued to be pervasive throughout the twentieth century.

9. On the rise of mathematical modernism, see Herbert Mehrtens, *Moderne Sprache Mathematik: Eine Geschichte des Streit um die Grundlagen der Disziplin und des Subjekts Formaler System* (Frankfurt: Suhrkamp, 1990); and Jeremy Gray, *Plato's Ghost: The Modernist Transformation of Mathematics* (Princeton, NJ: Princeton University Press, 2008).

10. The division was enforced on an institutional level by the separation of pure and applied mathematics across universities and technical schools. The separation on the technical level is much harder to define and became a source of heated debate throughout the twentieth century. Labeling a certain mathematical theory as "pure" or "applied" might have been difficult, but by the turn of the century pure and applied mathematics increasingly began to be acknowledged as two distinct professional identities. See Lewis Pyenson, *Neohumanism and the Persistence of Pure Mathematics in Wilhelmian Germany* (Philadelphia: American Philosophical Society, 1983); and Gert Schubring, "Pure and Applied Mathematics in Divergent Institutional Settings in Germany: The Role and Impact of Felix Klein," in *The History of Modern Mathematics: Institutions and Applications*, ed. David E. Rowe and John McCleary (Boston: Academic Press, 1989), 171–222.

11. Lorraine J. Daston and Peter Galison, *Objectivity* (Cambridge, MA: Zone Books, 2010), 254.

12. Leo Corry, *Modern Algebra and the Rise of Mathematical Structures* (Basel: Birkhäuser Verlag, 2004).

13. Leo Corry, "Mathematical Structures from Hilbert to Bourbaki: The Evolution of an Image of Mathematics," in *Changing Images in Mathematics: From the French Revolution to the New Millennium*, ed. Umberto Bottazzini and Amy Dahan-Dalmedico (London: Routledge, 2001), 167–85.

14. Mehrtens, "Mathematical Models," 278.

15. Morris Kline dates the beginning of this transformation as early as the eighteenth century. "After 1700, more and more notions, further removed from nature and springing full-blown from human minds, were to enter mathematics and be accepted with fewer qualms. For the genesis of its ideas mathematics gradually turned from the sensory to the intellectual faculties." Morris Kline, *Mathematical Thought from Ancient to Modern Times*, vol. 2 (New York: Oxford University Press, 1990), 393.

16. Amy Dahan-Dalmedico, "An Image Conflict in Mathematics after 1945," in *Changing Images in Mathematics*, 223–53; and Alma Steingart, "Conditional Inequalities: American Pure and Applied Mathematics, 1940–1975" (Ph.D. diss., Massachusetts Institute of Technology, 2013).

17. Marshall H. Stone, "Mathematics and the Future of Science," *Bulletin of the American Mathematical Society* 63, no. 2 (1957), 67.

18. Stone, 68.

19. This is not to say that until the 1970s mathematicians did not use illustrations or modeling in their work. Certain areas of mathematical research lend themselves more easily to visualization than others, and usually those mathematicians who were so inclined worked in specific areas. Rather, the point is that such an approach had been out of fashion, as it did not accord with the

prevailing philosophy that privileged abstractions and generalizations. See, for example, Siobhan Roberts, *King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry* (New York: Walker, 2006). On some of the earliest appropriations of computer graphics to mathematics, see Alma Steingart, "A Four-Dimensional Cinema: Computer Graphics, Higher Dimensions, and the Geometrical Imagination," in *Visualization in the Age of Computerisation*, ed. Annamaria Carusi, Aud Sissel-Hoel, and Timothy Webmoor (New York: Routledge, 2014), 170–96.

20. The reasons for this change are varied. By 1970, the strong division between pure and applied mathematics that characterized the growth of the field in the aftermath of World War II began to fall apart. Starting at the end of the 1960s, the public decree of scientific idealism and the call for utilitarian applications made mathematicians' call for autonomy and the independence of mathematical theories from the physical world at best suspect and at worst contemptuous. Further, the growing influence of computer science on mathematical practice questioned the view of mathematics as ideal contemplation and blurred the lines between the pure and the applied. Finally, the rise of new mathematical fields such as catastrophe theory and fractal geometry, with their emphasis on intuition and their rejection of the axiomatic, helped foster a recommitment to the concrete. David Aubin, "Forms of Explanation in the Catastrophe Theory of René Thom: Topology, Morphogenesis, and Structuralism," in *Growing Explanations: Historical Perspectives on Recent Science*, ed. M. Norton Wise (Durham, NC: Duke University Press, 2004), 95–130; Stephanie Dick, "AfterMath: The Work of Proof in the Age of Human–Machine Collaboration," *Isis* 102, no. 3 (1 September 2011): 494–505; and Steingart, "Conditional Inequalities."

21. An extensive literature in the philosophy of science attends to the role of models in scientific practice. Scholars have investigated the ontological status of models across multiple scientific disciplines, interpreting them variously as structures, idealizations, mediators, and fictions. The modeling practices I describe here (and which I denote as manifestations) are, however, different in kind. Unlike their scientific counterparts, they do not take their point of departure from the physical world, which is the domain of most scientific activity. Rather, they are models of mathematical theories. Since I am only interested in the ways in which practicing mathematicians generate understanding and create new mathematical knowledge, my position is independent of any particular mathematical philosophy. Thus, even a mathematician who adheres to a Platonistic view is required to grapple with the objects and theories she studies if she is to use them in her work. This grappling that extends beyond the symbolic is my focus in the present article. See also Nancy Cartwright, *How the Laws of Physics Lie* (Oxford, UK: Clarendon Press, 1983); Ronald N. Giere, *Explaining Science: A Cognitive Approach* (Chicago: University of Chicago Press, 1988); Mary S. Morgan and Margaret Morrison, *Models as Mediators: Perspectives on Natural and Social Science* (Cambridge, UK: Cambridge University Press, 1999); Roman Frigg and Matthew Hunter, *Beyond Mimesis and Convention: Representation in Art and Science* (Dordrecht, The Netherlands: Springer Science and Business Media, 2010); and Michael Weisberg, "Three Kinds of Idealization," *Journal of Philosophy* 104, no. 12 (2007): 639–59.

22. Historians of mathematics recently have begun calling attention to the material culture of mathematics, demonstrating how blackboards, computing devices, calculating machines, plaster and string models, as well as simple paper and pencil, condition and affect the work of mathematicians. Most of these studies, however, focus on how these objects are used in pedagogical work

or aid in theorem-proving. Michael J. Barany and Donald MacKenzie, "Chalk: Materials and Concepts in Mathematics Research," in *Representation in Scientific Practice Revisited*, ed. Catelijne Coopmans et al. (Cambridge: MIT Press, 2014), 107–29; Dick, 494–505; Peggy Aldrich Kidwell, "American Mathematics Viewed Objectively: The Case of Geometric Models," in *Vita Mathematica: Historical Research and Integration with Teaching*, ed. Ronald Calinger (Washington, DC: Mathematical Association of America, 1996), 197–208; Peggy Aldrich Kidwell, Amy Ackerberg-Hastings, and David Lindsay Roberts, *Tools of American Mathematics Teaching, 1800–2000* (Baltimore: Johns Hopkins University Press, 2008); Donald A. MacKenzie, *Mechanizing Proof: Computing, Risk, and Trust* (Cambridge: MIT Press, 2001); and Christopher Phillips, "Mirrors of the Mind: Chalkboards and the Practice of Mathematics" (paper presented at the annual meeting of the History of Science Society, Cleveland, 2011).

23. Brian Rotman advocates an analysis of mathematics as embodied knowledge. However, by positing a semiotic approach to mathematics according to which a single mathematician comprises three distinct agencies (the person, the subject, and the agent), Rotman primarily analyzes the way a mathematician's body figures within the written form of mathematical practice (e.g., symbolic notations of numbers and diagrams). Rotman's main concern is resurrecting the corporeality of the mathematician from written paper. In drawing attention to the bodies of mathematicians, I focus in this essay on the role of bodily perception in mathematical research beyond symbolic notation. As the history of the sphere eversion demonstrates, the symbolic realm that is opened up to mathematicians extends beyond a set of well-defined marks on paper. Moreover, the mathematicians about whom I write are not fragmented subjects as Rotman suggests. More recently, Michael Barany and Donald MacKenzie drew attention to the bodily practices of mathematicians through an analysis of the role of blackboards and chalk in mathematical practice. However, their investigation is devoted almost completely to inscription practices. I do not wish to underplay the dominant role of written practices for mathematicians but rather to suggest that overemphasizing the written often obscures other modes of mathematical engagement, ones that posit embodied rather than "virtual" mathematicians. What I am after here is precisely the ways in which these various capacities traffic between multiple kinds of mathematical signification. Barany and MacKenzie, "Chalk"; Brian Rotman, *Ad Infinitum: The Ghost in Turing's Machine* (Stanford: Stanford University Press, 1993); and Brian Rotman, *Mathematics as Sign: Writing, Imagining, Counting* (Stanford: Stanford University Press, 2000).

24. What follows is in no way a complete history of the problem. I describe in full only four attempts that are representative of distinct approaches mathematicians have taken over the past five decades. I do not, for example, discuss John Hugh's computational work, the film *Outside In*, illustrations by either Jean-Pierre Petit or George Francis, or the latest book by Carter Scott dedicated to yet another solution. J. Scott Carter, *An Excursion in Diagrammatic Algebra: Turning a Sphere from Red to Blue* (Singapore: World Scientific Publishing, 2011).

25. My use of haptic vision follows the distinction articulated by Alois Riegl. As Margaret Iversen notes, for Riegl the near or haptic mode of vision was "analogous to the sense of touch in the way that it must synthesize mentally a number of discontinuous sensory inputs." Margaret Iversen, *Alois Riegl: Art History and Theory* (Cambridge: MIT Press, 2003), 9.

26. The first mathematical treatise in English to use the title *Topology* came out in 1930, by

Solomon Lefschetz. Until then the more common name for the field was *analysis situs*. On the origin of topology, see Moritz Epple, "Topology, Matter, and Space, I: Topological Notions in 19th-Century Natural Philosophy," *Archive for History of Exact Sciences* 52, no. 4 (1998): 297–392.

27. Gray, 452.

28. During the first half of the twentieth century, the main division in the growth of the field, which was later unified, was between point-set topology and combinatorial or algebraic topology.

29. From a topological perspective the sphere is two-dimensional because its surface is equivalent to a two-dimensional plane (think about poking a hole into it and flattening the surface). Whereas a regular tennis ball is three dimensional, its outside surface, having no thickness, is two-dimensional.

30. Given two topological spaces (X and Y), two immersions (f and g) from X to Y are *homotopic* if there exists a continuous function F: X × [0,1]→Y, such that F(x,0) = f(x) and F(x,1) = g(x).

31. Nelson L. Max, *Turning a Sphere Inside Out: A Guide to the Film* (Chicago: International Film Bureau, 1976), 334.

32. Max, *Turning a Sphere Inside Out: A Guide*, 334.

33. Anthony Phillips, "Turning a Surface Inside Out," *Scientific American* 214, no. 5 (1 May 1966): 112.

34. Max, *Turning a Sphere Inside Out: A Guide*, 335; and Phillips, "Turning a Surface Inside Out," 112.

35. Partly, this conviction was supported by the fact that in the 1930s mathematician Hassler Whitney had proved the impossibility of everting a circle. Whitney proved that no procedure exists by which a circle on a plane (think of a two-sided rubber band) can be turned inside out without creating a corner at some point during the transformation. Extrapolating from the case of a one-dimensional line to a two-dimensional surface and considering the fact that no explicit solution had yet been found, mathematicians were therefore convinced of the impossibility of everting a sphere. Hassler Whitney, "On Regular Closed Curves in the Plane," *Compositio Mathematica* 4 (1937): 276–84.

36. Stephen Smale, "A Classification of Immersions of the Two-Sphere," *Transactions of the American Mathematical Society* 90 (1958): 281–90.

37. Max, *Turning a Sphere Inside Out*.

38. For example, every mathematician's firsthand account asserts that when Smale showed the result to his graduate adviser, Raoul Bott, the latter "asked to be shown an explicit geometrical construction of an eversion" and "warned him that there was an 'obvious counterexample.'" Phillips, "Turning a Surface Inside Out," 112; and George Francis and Bernard Morin, "Arnold Shapiro's Eversion of the Sphere," *Mathematical Intelligencer* 2, no. 4 (1980): 200.

39. George K. Francis, Tamara Munzner, and Andrew Hanson, "Interactive Methods for Visualizable Geometry," *IEEE Computer* 27, no. 7 (1994): 10.

40. The biographical information in this section regarding Anthony Phillips and his work on the sphere eversion is taken from a personal interview with the author. Anthony Phillips, interview by author, 16 May 2011.

41. The real projective plane is a compact nonorientable two-manifold; it is most commonly defined as the space of lines in three-dimensional Euclidean space that pass through the origin.

42. In topology, a covering map p of topological spaces p: $C \to X$ is a continuous map such that

for every point $x \in X$ there exists an open neighborhood Ux, such that the inverse image of U under p is the union of disjoint open sets in C, each of which is homeomorphic via p with U.

43. Interchanging the two sheets of the cover and repeating the process would then lead to the sphere turned inside out.

44. Phillips, interview.

45. Phillips, "Turning a Surface Inside Out," 112.

46. Phillips, interview.

47. Phillips, interview.

48. A paper detailing Shapiro's original idea was published only in 1980, more than two decades after Shapiro first explained the idea to his colleagues. Francis and Morin, "Arnold Shapiro's Eversion of the Sphere."

49. The biographical information regarding Max is taken from a personal interview. Nelson L. Max, interview by author, 17 February 2011.

50. The stated goal of the conference was to bring together mathematicians and physicists, who, according to the organizers, had drifted apart in recent decades.

51. Nelson L. Max, interview by author, 17 February 2011.

52. Allyn Jackson, "The World of Blind Mathematicians," *Notices of the American Mathematical Society* 49, no. 10 (2002): 1246–51.

53. James R. Griesemer, "Three-Dimensional Models in Philosophical Perspective," in *Models*, 440.

54. Levy, 31.

55. In 1979, Morin published an article with Petit in *Pour la science* introducing readers to the problem of the sphere eversion. Bernard Morin and Jean-Pierre Petit, "Le retournement de la sphère," *Pour la science* 15 (1979): 34–49. Like Phillips's earlier article, the verbal explanations were accompanied by numerous colorful illustrations drawn by Petit. The article captured the attention of Jacques Lacan, who invited the two mathematicians to a meeting. Morin, who apparently was averse to all psychoanalytic theories, refused the meeting. However, Petit was open to the idea, and he and Lacan met shortly after the publication of the article. Their initial meeting was soon followed by two more in which the conversation focused on Boy's surface, which Petit explained using drawings and cardboard models he built specifically for Lacan. For a conversation describing those meetings, see Jean-Pierre Petit and Fabrice Guyod, "Récit des trois rencontres entre Jean-Pierre Petit et Jacques Lacan, tournant autour de la surface du cross-cap et de la surface de Boy," *Figures de la psychanalyse* 14, no. 2 (2006): 181–204.

56. Morin's version is considered simpler than Phillips's because it includes a smaller number of "critical stages" or "topological events."

57. The main problem with clay, according to Max, is that it cannot be used to depict the internal structure of the sphere during the transformation. With clay Max had no way to record the movement of surfaces passing through one another.

58. The project produced four computer animations of mathematical problems. For Max's personal recollections, see Nelson L. Max, "Computer Animation in Mathematics, Science, and Art," in *Computers in Mathematics*, ed. David V. Chudnovsky and Richard D. Jenks (New York: Marcel Deckker, 1990), 321–45; and Nelson L. Max, "My Six Years to Evert a Sphere," *IEEE Annals*

of the History of Computing 20, no. 2 (1998): 42.

59. Charles Pugh, interview by author, 30 May 2011.

60. Morin's version had the property that halfway through the deformation, which is the point at which exactly half of the surface is turned inside and the other half turned outside, the surface has fourfold symmetry. The symmetry of the halfway model implies that, in order to complete the eversion, all one has to do is to rotate the model by ninety degrees and retrace the same steps. This effectively cut in half the work of modeling the eversion. If you could reach Morin's halfway point following a regular homotopy, then you could entirely evert the sphere.

61. The models disappeared in the middle of the night a few years later. Who took them is not known, and all that remains of them is their appearance in Max's movie.

62. These collections are on display in various departments of mathematics across the United States. However, they are no longer used for pedagogical purposes. Instead, they are exhibited for their aesthetic and historical value. For a list of collections of mathematical models around the world, see Angela Vierling-Claassen, "Collections of Mathematical Models," Angela Vierling-Claassen: Mathematics, Teaching, and Relationships [blog], n.d. [ca. 2013], http://angelavc.wordpress.com/collections-of-mathematical-models/.

63. Gerd Fischer, *Mathematical Models: From the Collections of Universities and Museums* (Braunschweig: Vieweg, 1986); Kidwell, "Viewed Objectively"; and Mehrtens, "Mathematical Models."

64. Some photographs from the series accompanied the publication of "Crise de l'objet" by André Breton in *Cahiers d'art* in 1936. The volume also included an article by Christian Zervos titled "Mathématique et art abstrait." For more on Ray's photographs, see Isabelle Fortuné, "Man Ray et les objets mathématiques," *Études photographiques*, no. 6 (1 May 1999), http://etudesphotographiques.revues.org/190.

65. Edwin Mullins, *Man Ray: A Life in the Day of Man Ray* (1991; Chicago: Home Vision, 1994), quoted in Angela Vierling-Claassen, "Models of Surfaces and Abstract Art in the Early 20th Century" (paper presented at the Bridges Conference 2010), http://bridgesmathart.org/2010/cdrom/proceedings/46/paper_46.pdf.

66. Pugh, interview.

67. The mathematicians who have worked on the problem repeatedly appeal to a language of mental imagery (e.g., "mental image," "mental picture") to describe their work. In explaining the motivation behind the production of the film *Inside Out*, mathematician Albert Marden writes, "as mathematicians we can often, by analogy and extrapolation from what we do experience, 'see' abstract objects in our mind's eye. . . . Computer graphics then allows us to share some of this vision, as if it too were part of our everyday world." Albert Marden, "Afterword: Why Outside In?" in *Making Waves*, 46–47. The rejection of visual thinking was concomitant with the rise of mathematical modernism and formalism. The various uses of mental imagery to which mathematicians appeal could all be loosely captured by the German *Anschauung*, which, following Kantian philosophy, had a fairly stable meaning in mathematics during the nineteenth century, "mainly as geometrical intuition in drawing conclusions from mental imagery in geometry and analysis." Yet with the rise of formalism, *Anschauung* no longer had a place in formal mathematical discourse because at least some mathematicians believed mathematics should be the realm of pure thought.

Herbert Mehrtens, "The Social System of Mathematics and National Socialism: A Survey," *Sociological Inquiry* 57, no. 2 (1 April 1987): 167.

68. Max saw *La faim* at a UAIDE (Users of Automatic Display Equipment) conference in the early 1970s.

69. For a technical report, see Nelson L. Max and William H. Clifford Jr., "Computer Animation of the Sphere Eversion," in *Proceedings of the 2nd Annual Conference on Computer Graphics and Interactive Techniques*, SIGGRAPH '75 (New York: ACM, 1975), 32–39.

70. Evans and Sutherland, who at the time headed the computer graphics center at the University of Utah, developed the machine for flight simulators.

71. Probably the best example of this transformation was the establishment of the Geometry Center at the University of Minnesota, whose stated goal was to bring computer graphics to bear on problems in pure mathematics. The center, which opened in 1991, had a short life, closing its doors in 1998. Some of the programs developed there, however, such as Geomview, are still in use. By the 1990s, proposals for a new field of mathematical visualization emerged, and while the constitution of the field remains unclear, a visualizable approach has been on the rise for the past three decades in certain areas of mathematics. See Francis, Munzner, and Hanson; and Richard Palais, "The Visualization of Mathematics: Towards a Mathematical Exploratorium," *Notices of the American Mathematical Society* 46 (1999): 647–58.

72. Even before computer graphics, mathematicians were able to produce continuous representations of mathematical theories and objects. Max had used Claymation for this purpose. The novelty of computer animation for mathematicians was the medium's ability to merge the computational with the visual. In 1969, Max explained that computer animation is the perfect tool with which to create films of moving curves and surfaces, since "it provides a direct link from the mathematical computations to the film, bypassing tedious hand plotting." Nelson Max, "Computer Animation for Mathematical Films," in *UAIDE Proceedings of the Eighth Annual Meeting*, 245–52.

73. The film is available online at https://www.youtube.com/watch?v=wO61D9x61NY.

74. William P. Thurston, "Making Waves: The Theory of Corrugations," in *Making Waves*, 40; emphasis added.

75. Thurston, "Making Waves," 40.

76. Max, interview. Phillips was similarly skeptical of his article's impact. Laughingly, he noted, "no one ever told me 'I read your article and I understand how you turn the sphere inside out.'" Phillips, interview.

77. George Francis et al., "The Minimax Sphere Eversion," in *Visualization and Mathematics: Experiments, Simulations and Environments*, ed. Hans-Christian Hege and Konrad Polthier (New York: Springer-Verlag, 1997), 3.

78. Figuring out how to work with the Surface Evolver to create the minimax eversion was not a straightforward task and involved repeated trial and error. Francis et al., "The Minimax Sphere Eversion." The Surface Evolver is in the public domain. See http://www.susqu.edu/brakke/evolver/evolver.html.

79. Christopher Kelty and Hannah Landecker, "A Theory of Animation: Cells, L-Systems, and Film," *Grey Room* 17 (2004): 32.

80. George K. Francis, Stuart Levy, and John Sullivan, "Making the Optiverse: A Mathematician's

Guide to AVN, a Real-Time Interactive Computer Animator," in *Mathematics, Art, Technology, Cinema*, ed. Michele Emmer and Mirella Manaresi (Berlin: Springer, 2003), 42.

81. George Francis and John M. Sullivan, "Visualizing a Sphere Eversion," *IEEE Transactions on Visualization and Computer Graphics* 10, no. 5 (2004): 511.

82. Francis and Sullivan, "Visualizing a Sphere Eversion," 514.

83. For more on virtual environments in scientific practice, see Natasha Myers and Joseph Dumit, "Haptics: Haptic Creativity and the Mid-Embodiments of Experimental Life," in *A Companion to the Anthropology of the Body and Embodiment*, ed. Frances E. Mascia-Lees (Oxford, UK: Wiley-Blackwell, 2011), 239–61.

84. Francis and Sullivan, "Visualizing a Sphere Eversion," 513.

85. Francis and Sullivan, "Visualizing a Sphere Eversion," 515.

86. Phillips, interview.

87. Maurice Merleau-Ponty, *The Visible and the Invisible*, trans. Claude Lefort (Evanston, IL: Northwestern University Press, 1968), 210. Originally published in French as Maurice Merleau-Ponty, *Le visible et l'invisible* (Paris: Gallimard, 1964).

88. Merleau-Ponty, *The Visible and the Invisible*, 213–14.

89. As Eric de Bruyn shows, in the 1960s topology served as a potent metaphor across a wide range of disciplines. Topological modes of thought functioned both discursively and materially in artistic practice precisely by offering a new means of rethinking non-Euclidean spatial relations. Eric de Bruyn, "Topological Pathways of Post-Minimalism," *Grey Room* 25 (2006): 32–63.

90. I distinguish the notion of *manifestation* from the more common *representation* in scientific practice. The rich and fruitful literature that has emerged around the study of scientific representation over the past two decades has shed light on a wide range of concepts from objectivity to modeling to mediation. Yet, as the editors of *Representation in Scientific Practice Revisited* note in the introduction to their volume, despite the evident usefulness of the term, it carries much "philosophical baggage"—namely, its function within a theory of referents. As Lorraine Daston, who contributed to the volume, writes in her essay, "whatever the philosophical standpoint, representation is always derivative from some presentation, and therefore directs attention toward how rather than what we know, epistemology rather than ontology." Lorraine Daston, "Beyond Representation," in *Representation in Scientific Practice Revisited*, 320.

91. Building upon the use of transduction in structural biology, Natasha Myers argues that the term can also describe the work of scientists themselves. "Through their interactions with each other and with their models, protein modelers can be seen to transduce and so propagate the molecular affects and gestures they have cultivated in order to communicate their feeling for protein forms and mechanism." Natasha Myers, "Animating Mechanism: Animations and the Propagation of Affect in the Lively Arts of Protein Modeling," *Science Studies* 19, no. 2 (2006): 23.