Bourbaki: A Secret Society of Mathematicians
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Towards Axioms and Structures

An intellectual construction endowed with profound unity, a hierarchy of abstract structures built on a foundation of axioms: this is how Bourbaki viewed mathematics. The group's beliefs gained many followers.

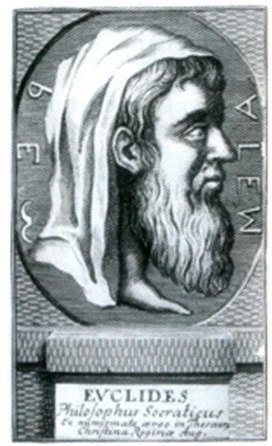
Nicolas Bourbaki did much more than write a voluminous and ambitious mathematical treatise. During its golden years of the sixties and seventies, the group also propagated a certain vision of mathematics. This new ideology was adopted by many mathematicians around the world, and it even influenced some fields outside of mathematics. Remember that Bourbaki's global views of mathematics developed over the course of the group's first ten of fifteen

The ritual tea hour at the Institut des Hautes Études Scientifiques at four o'clock. In the foreground are Jean Dieudonné, Alexandre Grothendieck (from behind), Michel Demazure, and François Bruhat.

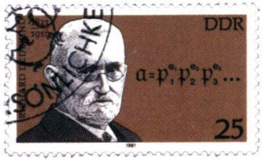
years, as its members completed the outline of their treatise and started to write the first volumes. Also remember that the views held by Bourbaki as a group are not necessarily the same as the those held by its individual members. While some of the Bourbaki members propagated the Bourbakian ideology quite faithfully, Bourbaki is heterogeneous like any group of humans. Opinions varied significantly from one member to another, and it is a job for historians to sort out these various opinions. Given the secrecy and collective nature of Bourbaki, this job remains a hard one to this day. Finally, it is important to remember that the times and the members have changed, and so the Bourbakian vision of today is probably not the same as that of the fifties. However, as the Bourbaki of today does not express its vision of mathematics (if it has one at all), we must settle for studying the vision held in the past.

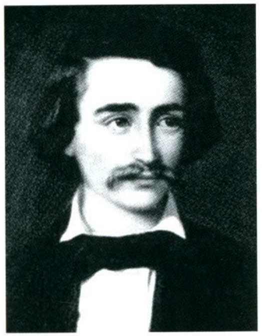
Bourbaki spread its conception of mathematics through various texts and lectures. One of the most important of these is the article *L'architecture des mathématiques* ("The Structure of Mathematics") published in 1947 and signed by Nicolas Bourbaki himself. Although this article is a real manifesto of Bourbaki's ideology, it appears that the group didn't discuss the article much before its publication. *L'architecture des mathématiques* was probably written by Dieudonné, who usually didn't express himself subtly. Discussing this article in his book *Mathématique: (récit)*, the author-mathematician Jacques Roubaud writes that "Bourbaki calmy wields Neanderthalian clubs of philosophy, in contrast to his habitual rabbit-like caution."

So how did Bourbaki see mathematics? Its philosophy revolves around three key notions: the unity of mathematics, the axiomatic method, and the study of structures. Today the unity of mathematics has become a well-worn topic that comes up every time mathematicians take a slightly global look at mathematics. Mathematicians are always emphasizing that geometry, algebra, analysis, and number theory are no longer separate topics; that modern mathematical research applies to all domains; that, for example, the proof of a theorem in number theory uses a mixture of concepts and methods from analysis, geometry, and algebra. Yet this unity was not so blatantly obvious to mathematicians of the thirties and forties, the era when Bourbaki formed. In *L'architecture des mathématiques*, Bourbaki writes that "one can ask himself whether this extravagant proliferation [of mathematical results] is the product of a vigorous organism that acquires more cohesion and unity with every addition it receives, or whether, on the contrary, if it is the outwards manifestation of an increasing fragmentation due to the very nature of mathematics, and whether mathematics is becoming a Tower of Babel of autonomous disciplines,



Euclid developed a precursor to the axiomatic method.





A stamp and portrait of Richard Dedekind (1831–1916). Dekedind was one of the fathers of modern algebra.

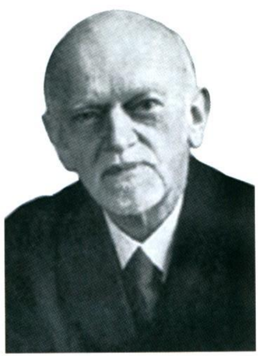
isolated from one another in their goals, methods, and even in their language. In short, is modern mathematics a single *mathématique* or many *mathématiques*?"

Bourbaki's reply to this question is clearly in favor of the singular *mathématique*, which the group uses in the title of its treatise, *Éléments de mathématique*. In the eyes of Bourbaki, this unity is strong, in some respects even stronger than it is considered today. As Bourbaki writes, "We believe that the internal evolution of mathematical sciences, despite appearances, has strengthened the unity of its various subjects more than ever. It has created a sort of nucleus that is more cohesive than ever before. The most important aspect of this evolution is the systematization of the connections between the various areas of mathematics. This is known as the axiomatic method."

The Axiomatic Method According to Hilbert

What exactly is the axiomatic method? To begin with, an axiom is often a self-evident principle whose validity is accepted without proof. An axiom is a sort of fundamental truth, but an axiom can also be an invented property or rule that has no immediate connection to reality. An axiomatic theory starts with the definitions of the objects it will deal with. Then axioms (sometimes called postulates) that the objects in question must obey are added. Finally, logical chains of reasoning, whose validity can be verified without reference to intuition or experience, lead from the objects and axioms to less obvious properties, called theorems.

The most ancient example of the axiomatic method is found in Euclid's geometry. In his *Elements*, Euclid of Alexandria presents geometry by starting with the fundamental objects of a point ("that which has no part"), a line ("a length without thickness"), a curve, a plane, and so forth. He proceeds by stating five axioms: 1) Given any two points, a line segment exists that joins them; 2) A line segment can be extended indefinitely; 3) A circle can be constructed with any given center and radius; 4) All right angles are equal; and 5) If a line crosses any two lines such that the sum of the interior angles is less than two right angles, then the two lines cross when extended on the corresponding side. The fifth axiom is equivalent to the famous parallel postulate ("Given a line and point not on the line, there exists a unique line passing through the point and parallel to the first line"), which has made so many mathematical minds spin.



David Hilbert (1862–1943).

From Bourbaki to Oulipo, Lévi-Strauss, and Piaget

Raymond Queneau (1903–1976) in 1951.

In the fifties and sixties, Bourbaki's vision of mathematics influenced areas outside of mathematics. In particular, it expressed itself in literature with Oulipo (short for *Ouvroir de littérature potentielle*, meaning "Workshop for Potential Literature"), a movement related to surrealism. Created in 1960 by Raymond Queneau and François le Lionnais, Oulipo aimed to explore new forms of literature obtained by imposing constraints ("structures") of a mathematical nature. For example, Queneau wrote a literary translation of Hilbert's *Foundations of Geometry*, which he titled *Les fondements de la littérature d'après David Hilbert* ("The Foundations of literature according to David Hilbert"). In this work, Queneau imitates the axioms of geometry developed by Hilbert by replacing the words "points," "lines," and "planes" by "words," "sentences," and "paragraphs," respectively. The results of such exercises are often hilarious. Oulipo, which included some mathematicians (notably Jacques Roubaud and Claude Berge), had direct contacts with Bourbaki. For example, Queneau attended a Bourbaki conference in 1962. Their humor, taste for secrecy, and use of structures helped bring the two groups together.

Bourbaki's structuralism was also fashionable in French social sciences. Structuralism had been developing in the social sciences since the fifties, following the publication of *Structures élémentaires* by the anthropologist Claude Lévi-Strauss. Lévi-Strauss had met André Weil in 1943 in New York, which led to a small collaboration: by using group theory, André Weil solved a combinatorial problem about marriage rules in an Australian tribe. This contribution appeared in an appendix of Lévi-Strauss's book. Another cross-disciplinary encounter was that of Jean Dieudonné with Jean Piaget during a 1952 conference on mathematical and mental structures. This led Piaget to consider the existence of a direct connection between the structures in a child's mental processes and the mathematical mother-structures.

While Bourbaki's structures were often mentioned in social science conferences and publications of the era, it seems that they didn't play a real role in the development of these disciplines. David Aubin, a science historian who analyzed Bourbaki's role in the structuralist movement in France, believes Bourbaki's role was that of a "cultural connector." According to Aubin, while Bourbaki didn't have any mission outside of mathematics, the group represented a sort of link between the various cultural movements of the time. Bourbaki provided a simple and relatively precise definition of concepts and structures, which philosophers and social sciences believed was fundamental both within their disciplines and in bridges among different areas of knowledge. Despite the superficial nature of these links, the various schools of structuralist thinking, including Bourbaki, were able to support each other. So, it is not a coincidence that these schools suffered a simultaneous decline in the late 1960s.

An extract from *Les fondements de la littérature d'après Hilbert* by Raymond Queneau. Published in *La bibliothèque Oulipienne, présenté par Jacques Roubaud* (Slatkine).

Group I (Incidence Axioms)

I, 1 *There exists a sentence containing any two given words.*
Commentary: Obvious. Example: Consider the two words "A" and "A." There is a sentence containing these two words: "A violinist gave an A to the singer."

I, 2 *There exists at most one sentence containing any two given words.*
Commentary: This, on the other hand, is surprising.
But let's consider two words such as "long" and "bed." It is clear that once the sentence containing them has been written, namely, "For a long time I would go to bed early," any other expression like "for a long time I would retire to bed early" or "for a long time I never went to bed late" is a pseudo-sentence that must be rejected in accordance with this axiom.
Scholium: Naturally, if one wrote, "For a long time I would retire to bed early," it is the sentence, "For a long time I would go to bed early" that must be rejected in accordance with axiom I.2. In other words, *In Search of Lost Time* cannot be written twice[1].

I, 3 *Any sentence contains at least two words. There exist at least three words not all belonging to the same sentence.*
Commentary: Thus there are no sentences consisting of a single word. "Yes," "No," "Hey," and "Psst" are not sentences. As for the second part of the axiom: one thus supposes that the language in question contains at least three words (this is trivial for the case of French) and that no sentence can contain *all* the words of the language (or all but one, or all but two).

I, 4a *There exists a paragraph containing three given words not all belonging to the same sentence.*
Commentary: This immediately implies that a paragraph must contain at least two sentences.

One will note that the statements of axioms I, 1 through I, 4 do not comply to axiom I,2 because all four require the words "words" and "sentences" for their formulation, while this axiom states that there can only be a single sentence containing these two words.
This yields the following axiom of metaliterature: *Axioms do not obey the axioms.*

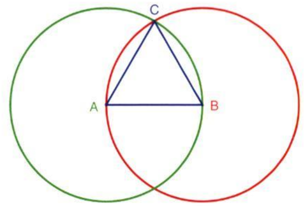
I, 4b *Every paragraph contains at least one word.*
Commentary: "Yes," "No," "Hey," and "Psst," which are not sentences according to I, 3, do not constitute paragraphs by themselves.

I, 5 *There exists at most one paragraph containing three given words not all belonging to the same sentence.*
Commentary: Like I, 2, this axiom is an axiom of uniqueness, the uniqueness of paragraphs in this case. In other words, if three words not all belonging to the same sentence are written in one paragraph, one cannot reuse these words in another paragraph. But, one might protest, what if they all belonged to the same sentence in the second paragraph? This axiom excludes this possibility.

I, 6 *If two words in a sentence belong to a paragraph, then all the words in that sentence belong to that paragraph.*
Commentary: No commentary necessary. [...]

[1] *Note that the opening line of Swann's Way, the first volume of Marcel Proust's In Search of Lost Time (Trans. C.K. Scott Moncrieff and Terence Kilmartin), is "For a long time I would go to bed early." Translator.*

The important thing here is not the content of Euclid's axioms but the fact that Euclid founded a theory on axioms that he could use (or more accurately, believed he could use) to derive geometric constructions and properties, such as the Pythagorean Theorem, rigorously. In reality, Euclid did not always use flawless rigor. He unwittingly used many properties that were suggested by visual intuition rather than included in the axioms or proved rigorously. A perfect example of this is his first proposition, which consists of constructing an equilateral triangle with a given side AB. To do this, Euclid draws two circles of radius AB, one centered at A and the other centered at B. He then proves that triangle ABC, where C is one of the two points where the two circle intersect, is equilateral. However, to prove this Euclid uses the property that the two circles must intersect in at least one point. While this is visually obvious, Euclid should have justified this statement rigorously.



When A and B are the centers of the circles, the triangle ABC is equilateral.

Despite this, what Euclid had developed was an important precursor to the axiomatic method. Moreover, it is not at all easy to come up with a coherent system of axioms for Euclidean geometry. In fact, this was achieved only a century ago, primarily by the great German mathematician David Hilbert in his book *Grundlagen der Geometrie* ("Foundations of Geometry"), published in 1899. Hilbert's project did far more than repair the flaws in Euclid's axioms: it was a stamp of approval for the modern axiomatic method in general.

What distinguishes the modern axiomatic method advocated by Bourbaki from Euclid's method? One main difference lies in its formal nature: the modern axiomatic method does not try to define basic concepts (like points and lines) that the theory in question will discuss. These basic concepts are treated as abstract entities whose nature and concrete meaning are insignificant. To illustrate this, Hilbert jokes that the "points," "lines," and "planes" used in the axioms of geometry could just as well be called "chairs," "tables," and "beer bottles." Only the relations between the fundamental entities defined by the axioms are important. The properties deduced from such a formal theory are completely general, since they could apply to a very different set of objects, as long as the axioms for that set of objects are the same. Later we'll see the simple example of how axioms are used to define a structure called a group.

It's important to understand that mathematicians do not construct sets of axioms directly out of nothing. The mathematician starts by studying a certain set of objects, and then he develops a set of axioms based on these objects. As Henri Cartan (one of the founders of Bourbaki) explained in a 1958 lecture in Germany, "A mathematician who sets about proving a theorem has in mind certain well-defined

1. What is a Group?

A non-empty set G is called a group if it is equipped with a composition law, denoted for example by $*$, that associates to each pair (x, y) of elements in G an element denoted by $x*y$ (also in G) and satisfies the following properties.

1. Associativity: for any x, y, z in G, $x*(y*z) = (x*y)*z$.
2. Existence of an identity element: there exists an element e of G such that $e*x = x*e = x$ for any x in G.
3. Existence of an inverse for any element: for each x in G, there exists an element x^{-1} such that $x*x^{-1} = x^{-1}*x = e$.

If the law $*$ is commutative—that is, if $x*y = y*x$ for any x and y in G—then we say that G is an abelian group. It is easy to deduce numerous properties from the above axioms. As an example, let's prove that if $x*y = x*z$, then $y = z$. Multiplying the both sides of the equality $x*y = x*z$ by x^{-1} on the left yields $x^{-1}*(x*y) = x^{-1}*(x*z)$. Then, by using axiom 1 followed by axiom 3, we obtain $e*y = e*z$. Finally, axiom 2 implies that this is equivalent to $y = z$. QED. Almost identical reasoning shows that the identity element is unique: if $x*e = x*e' = x$ for every x in G, then $e = e'$.

mathematical objects that he is studying at the moment. When he thinks he has found a proof, he starts carefully checking all his conclusions, and he realizes that only a very small number of the properties of the objects in question played any sort of role in the proof. He thus discovers that he could use the same proof for other objects having only the properties he needed to use. This demonstrates the simple underlying idea of the axiomatic method: instead of announcing which objects must be considered, it suffices to provide a list of properties [...] to be used in the investigation. Then one expresses these properties as axioms. From then on, it's no longer important to explain what objects are being studied. Instead, one can construct the proof so that it is true for any objects satisfying the axioms. It's quite remarkable that the systematic implementation of such a simple idea shook mathematics so thoroughly."

For Bourbaki, the axiomatic method is inseparable from the study of structures, the third key element of the group's vision of mathematics. What does Bourbaki mean by mathematical structures? As *L'architecture des mathématiques* explains, one starts with a set "of elements whose properties are not specified. Then one or several relations among these elements are added [...], and postulates are added that the given relation or relations must satisfy. These are the axioms of the structure in mind. Making an axiomatic theory out of a given structure is deducing logical consequences of the structure's axioms without using any other assumptions about the elements in question (and, in particular, without using any assumptions about their 'nature')."

Three Important Types of Structures

Let's illustrate the notion of structures with one of the most widespread and important mathematical structures: the group. The axioms that define groups are given in box 1. There are many concrete realizations of this abstract entity. Here are three, suggested by *L'architecture des mathématiques*:

1. The set of real numbers with the operation of ordinary addition.
2. The set of integers 1, 2, 3, 4, 5, 6 equipped with multiplication modulo 7. (The result of multiplication of two integers m and n modulo 7 is the remainder upon dividing the product mn by 7. For example, the result of 4 times 5 modulo 7 is 6, because 6 is the remainder when 20 is divided by 7.)
3. The set of translations of the Euclidean plane.

At the Chançay conference in 1936: Claude Chevalley (from behind), Szolem Mandelbrojt, Jean Delsarte, Jean Dieudonné, and André Weil (standing).

2. What Is a Ring? What Is an Ideal?

A ring is a set whose elements can be added, subtracted, and multiplied (but not necessarily divided). More precisely, a ring is a non-empty subset R equipped with an addition law $+$ and a multiplication law $*$ such that

1. R is an abelian group with the operation $+$. (The identity element for addition is usually denoted by 0.)
2. The law $*$ is associative and has an identity element (usually denoted by 1); that is, $x*(y*z)=(x*y)*z$ and $1*x=x*1=x$ for any x, y, and z in R.
3. The law $*$ distributes over $+$; that is, $x*(y+z)=x*y+x*z$ and $(x+y)*z=x*z+y*z$ for any x, y, and z in R.

The ring R is called commutative when both the addition and multiplication laws are commutative. A familiar example of a commutative ring is the set $\mathbf{Z}$ of integers, equipped with the usual operations $+$ and $*$ of addition and multiplication. Another example of a commutative ring is the set of polynomials in one variable with coefficients in the real numbers (these are expressions of the form $a_0+a_1x+a_2x^2+\cdots+a_nx^n$, where n is a positive integer called the degree of the polynomial and $a_1, a_2, a_3, \ldots, a_n$ are given real numbers called the coefficients), again equipped with the usual laws of polynomial addition and multiplication. An example of a non-commutative ring is the set of 2x2 matrices with real entries (these are tables of four numbers arranged in two lines of two elements each), which have many applications, including the representation of linear transformations acting on vector spaces. In this case, the addition and multiplication laws are the usual laws of addition and multiplication for matrices; see the diagram below.

Given a ring R, now consider a subset I of R. This subset I is a left ideal if the following properties are satisfied:

1. I is a subgroup of R with respect to the addition law $+$.
2. For any a in R and x in I, the product $a*x$ is an element of I.

The definition of a right ideal is analogous; just replace $a*x$ by $x*a$ in property 2. For example, the even numbers form an ideal (both left and right) of the ring $\mathbf{Z}$ of integers, because any multiple of an even number is even.

$$A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad B=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \qquad A+B=\begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix}=B+A$$

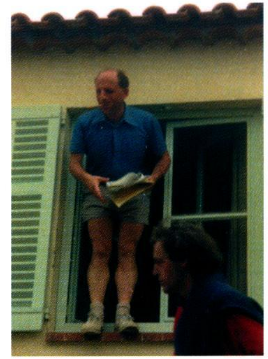
$$AB=\begin{pmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_3b_1+a_4b_3 & a_3b_2+a_4b_4 \end{pmatrix} \qquad BA=\begin{pmatrix} b_1a_1+b_2a_3 & b_1a_2+b_2a_4 \\ b_3a_1+b_4a_3 & b_3a_2+b_4a_4 \end{pmatrix}\neq AB$$

With a little high school math, it is not too hard to check that these three structures indeed satisfy the group axioms. For the first example, this is nearly obvious: any real x, y, and z satisfy $(x+y)+z=x+(y+z)$, the identity element is 0 since $x+0=0+x=x$, and the additive inverse of x is $-x$ since $x+(-x)=0$. For the second example, the identity element is the integer 1 and the inverses of 1, 2, 3, 4, 5, and 6 are respectively 1, 4, 5, 2, 3, and 6. For the third example, the identity element is the identity translation, which fixes every point in the plane, and each translation has an inverse that reverses the effect of that translation on the points of the plane.

Bourbaki differentiates three main types of structures. The structures involving a rule that associates any pair of elements to a third element are called algebraic structures. These sort of structures include groups, of course, as well as rings, ideals, fields, vector spaces, and so forth. The second type of structure involves an order (see box 4), which is a relation that orders, or compares, the elements of a set (although not necessarily all the elements). The structures of the third type are topological structures, which provide an abstract mathematical formulation of the intuitive concepts of *neighborhoods*, *limits*, and *continuity*. The next chapter discusses some structures of this type.

With the axiomatic method and these three main types of structures—algebraic structures, ordered structures, and topological structures—as guides, Bourbaki paints a picture of a mathematical universe where the organizing principle is a hierarchy of structures progressing from simple to complex and from general to specific. Bourbaki explains that "in the nucleus are the main types of structures [...], the mother-structures, one could say. Each of these types already contains quite a wide variety of objects, since one must differentiate the most general structure, the one with the least axioms, from the ones obtained by adding axioms, which each yield a crop of new results [...]. Beyond this innermost nucleus are the structures that one could describe as complex. These involve one or more mother-structures that aren't merely juxtaposed [...] but *combined* organically by one or more axioms connecting them [...]. Finally, farther out are the actual theories, where the elements of the sets in question, until now left undefined within the general structures, gain individual characteristics. This is where the classical mathematical theories are found." According to Bourbaki, individual theories of mathematics such as function theory, differential geometry, algebraic geometry, and number theory have lost their old autonomy to become "crossroads where the most general mathematical structures come together to act upon one another."

Having presented this vision of mathematics, the author of *L'architecture des mathématiques* emphasizes that it is merely a schematic



Pierre Cartier and Bernard Teissier in la Messuguière (near Grasse) in July 1975.

Charles Ehresmann (1905–1979)

and idealized view, and thus that it "must be treated only as a very rough approximation of the current state of mathematics."

The unity of mathematics, the axiomatic method, and the study of structures are not Bourbaki's own inventions. Mathematicians have always wondered whether or not mathematics is a single, unified subject. For instance, the question arises in simply trying to understand the connections between algebra and geometry (for example, why can a set of consecutive real numbers be identified with the points on a line segment?). The axiomatic method, pioneered by Euclid, entered modern mathematics at the end of the nineteenth century (with Hilbert's work, for example, and Richard Dedekind and Giuseppe Peano's axiomatization of integer arithmetic). As for the study of structures, at least of algebraic structures, Bourbaki was strongly influenced by van der Waerden's *Moderne Algebra*, which represented German mathematics during the years 1900–1930.

Bourbaki's role in these issues was to emphasize these three notions, to connect them, to try to extend the concept of structure emerging in the work of the German algebraists to all of mathematics. However, the Bourbakian vision of mathematics is not a perfectly constructed and coherent theory. Moreover, Bourbaki does not always follow its own philosophy of mathematics in *Éléments de mathématique.*

A working seminar at the Institut des Hautes Études Scientifiques. Grothendieck is presenting, and the first row contains (from left to right) Michel Demazure, François Bruhat, Pierre Samuel, and Jean Dieudonné (near the window).

3. What Is a Field?

A field is a set whose elements can be added, subtracted, multiplied, or divided (expecting division by zero, which is not defined). That is, a set K equipped with an addition law $+$ and a multiplication law $*$ is a field if

1. K is a ring under $+$ and $*$ and contains at least two elements.
2. Any nonzero element x of K has a multiplicative inverse x^{-1} (that is, x^{-1} satisfies $x * x^{-1} = x^{-1} * x = 1$). The zero element, which has no inverse, is by definition the identity element for addition.

Examples of fields are the set $\mathbf{Q}$ of rational numbers (numbers of the form p/q, where p and q are integers); the set $\mathbf{C}$ of complex numbers (the numbers of the form $x + iy$, where x and y are real and i satisfies $i^2 = -1$); and the subfield of $\mathbf{R}$ containing real numbers of the form $a + b\sqrt{3}$, where a and b are rational numbers.

Bourbaki the Ostrich

One of the main illustrations of this weakness is Bourbaki's attitude towards the axiomatization of set theory and, more globally, towards questions about the foundations of mathematics. Obtaining a satisfactory system of axioms for set theory, upon which the turn-of-the-century mathematicians wished to build all of mathematics, proved to be an arduous task that led logicians and mathematicians to much research on the foundations of mathematics. Among other things, they tried to prove that the mathematics resulting from these axioms was coherent, that the axioms could never lead to any sort of contradiction. This program, for a large part initiated by Hilbert, led to results that were surprising and hard to swallow. This is particularly the case of Kurt Gödel's proof that it is impossible to prove that a given system of axioms gives rise to a coherent system of mathematics by using only those axioms.

Faced with a crisis that was affecting mathematicians in the first third of the twentieth century, Bourbaki decided to hide its head in the sand and treat these metamathematical problems, so important to logicians, as uninteresting to mathematicians. However, it is hard to believe that the logical coherence of an axiomatic theory would be unimportant to a mathematician who, like Bourbaki, believed so much in the axiomatic method. Bourbaki's somewhat schizophrenic attitude—which is held by most mathematicians not working directly on the foundations of their discipline—materializes in the group's book

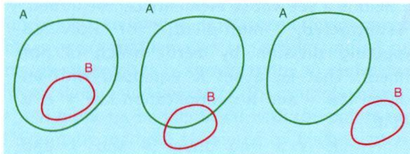
Saunders Mac Lane (1909–2005) was one of the founders of category theory.

4. What Is an Order?

The concept of an order generalizes common tools for comparison like "greater than or equal to" and "less than or equal to." More precisely, an order is a relation $\mathcal{R}$ on a set E that satisfies the following three properties.

1. Reflexivity: For any x in E, $x\mathcal{R}x$.
2. Transitivity: For any x, y, and z in E, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.
3. Antisymmetry: For any x and y in E, if $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x = y$.

The most well-known example of an order is the relation "greater than or equal to" on the set of real numbers. To check this, it suffices to check that the three order axioms are satisfied when $\mathcal{R}$ is replaced by the symbol $\geq$. Another example of an order is the relation of inclusion defined on the subsets of a given set E. We say that a set A is included in B if all the elements of A are also elements of B; this is written as $A \subset B$. It is simple to check that the relation $\subset$ satisfies the three order axioms; for example, if $A \subset B$



and $B \subset A$, then A and B must be the same set. Unlike the first example, where any two real numbers can be compared by the relation $\geq$ (such an order is called a total order), two sets A and B cannot necessarily be compared by the relation of inclusion. If A and B are disjoint or share some elements but not others, then neither $A \subset B$ nor $B \subset A$ is true (such an inclusion is called a partial order). A third example of an order is the relation of division defined on the set of integers (an integer a divides the integer b if b/a is an integer). This is also a partial order.

on set theory in *Éléments de mathématique.* This book met severe criticism, especially from logicians, for its narrow perspective and failure to address fundamental questions (see chapter 9).

Categories vs. Bourbakian Structures

Although Bourbaki's treatise contains numerous examples of structures, it leaves the concept of structures somewhat vague. Van de Waerden's book *Moderne Algebra,* which presented algebra as a hierarchy of structures, hadn't formalized this concept. "Van der Waerden didn't find it necessary to give an explicit explanation, be it formal or not, of what 'algebraic structures' and the 'structural study of algebra' really are," Leo Corry, a historian at the University of Tel Aviv, explains in

his book. Bourbaki, on the other hand, does provide a formalization of the concept of structures in the fourth chapter of the book on set theory. However, the following chapters of the treatise do not make use of this formalization—despite the fact that the treatise's first part, consisting of the first six books, was initially called "Fundamental structures of analysis." Corry's analysis of the concept of algebraic structure as it developed since the nineteenth century "suggests that [Bourbaki's] *Théorie des ensembles* [*Theory of Sets*], and in particular the concept of *structure* it defines, are not essential to the rest of *Éléments.* It is possible to read and understand every book of Bourbaki's treatise without having learned about the concept of *structures.* In principle, *Théorie des ensembles* could have been left out of the treatise, since it has neither heuristic value nor logical importance for any of the theories discussed in the treatise's later volumes, where the real importance of the treatise lies." Also, while the treatise's goal is to provide basic tools for mathematicians, "the concept of *structures* seems forced and unnatural." Pierre Cartier, a former member of Bourbaki, agrees with this view, affirming in a study of structuralism in mathematics that "our conclusion is final: Bourbaki did not produce a mathematical theory of structures, and perhaps did not want to do so."

Jean-Louis Koszul (born 1921) during a Bourbaki conference.

Today, no discussion of mathematical structures is complete without a discussion of category theory. Introduced around 1942 by Samuel Eilenberg (who would later become a member of Bourbaki) and Saunders MacLane, category theory provides an abstract and general framework for describing numerous mathematical situations and the connections between them. Things become technical and abstract very quickly, but we can give a rough description of what category theory entails. A category is given by a class of objects and, for every pair (A, B) of objects, a set of correspondances (called morphisms) from A to B. For example, in the category of sets the objects are sets and the morphisms from A to B are all the possible functions defined from A to B. In the category of groups, the objects are groups and the morphisms from a group A to a group B are all the homomorphisms from A to B. (A homomorphism f is a function preserving the structure of the group operation; that is, a function f is a homomorphism if for all x and y in A, $f(x * y) = f(x) \circ f(y)$, where $*$ is the composition law of the group A and $\circ$ is the composition law of the group B.) In addition, category theory studies the mapping between two categories, which are called functors.

The language of categories and functors spread rapidly during the sixties. Some Bourbaki members put it to great use, including Eilenberg of course, but also Charles Ehresmann and especially Alexandre Grothendieck. Category theory, which is much more general than the structures described by Bourbaki in *Éléments de mathématique*, could have played an important part in the structural vision of mathematics, but Bourbaki did not update its *Architecture des mathématiques*. More importantly, the group did not manage to use categories in its treatise, despite the group's numerous discussions and preliminary drafts on the subject. One of the reasons for this is that the task would have required a profound revision of the existing volumes. According to Pierre Cartier, "Bourbaki got away with talking about categories without really talking about them. If they were to redo the treatise, they would have to start with category theory. But there are still unresolved problems about reconciling category theory and set theory."

In her thesis on Bourbaki, Judith Friedman collected these remarks from Claude Chevalley, one of the group's founders, which relate to the idea that Bourbaki has become mathematically more bourgeois over the years: "From this point of view, category theory was more true to the spirit of Bourbaki than their theory based on structures—it was more structuralist! [...] In some sense, Bourbaki's rejection of categories was one of the most significant points in the transformation of the group's spirit. For the first time, something that people knew to be eminently Bourbakian was mostly rejected out of a desire to advance without addressing [...] the starting point." To try to rectify this, Chevalley wrote a book on category theory in the 1960s. However, the book was never published—the publishers, Hermann, lost the manuscript!

The fifties and sixties were the years of structuralism. The Bourbakian vision, which emphasized axioms and structures, was emulated by mathematicians as well as authors (particularly the Oulipo movement) and scientists in areas such as anthropology and psychology. Many criticized Bourbaki's vision of the structure of mathematics, and it is clear that their vision cannot encompass all the mathematical activity of today (see chapter 11). Nevertheless, this vision certainly encompassed the mathematics of their time and brought some clarity to the era in which it was conceived and expressed.

The logician Kurt Gödel and the physicist Albert Einstein at Princeton in 1950.